A non-equilibrium model for soil heating and moisture transport during extreme surface heating

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Abstract. With increasing use of prescribed fire by land managers and increasing likelihood of wildfires due to climate change comes the need to improve modeling capability of extreme heating of soils during fires. This issue is addressed here by developing a one-dimensional non-equilibrium (liquid-vapor phase change) model of soil evaporation and transport of heat, soil moisture, and water vapor, for use with surface forcing ranging from daily solar cycles to extreme conditions encountered during fires. The model employs a linearized Crank-Nicolson scheme for the conservation equations of energy and mass and its performance is evaluated against dynamic soil temperature and moisture observations obtained during laboratory experiments on soil samples exposed to surface heat fluxes ranging between 10,000 and 50,000 Wm$^{-2}$. The Hertz-Knudsen equation is the basis for constructing the model’s non-equilibrium evaporative source term. Some unusual aspects of the model that were found to be extremely important to the model’s performance include: (1) a dynamic (temperature and moisture potential dependent) condensation coefficient associated with the evaporative source term, (2) an infrared radiation component to the soil’s thermal conductivity, (3) a dynamic residual soil moisture, which is parameterized as a function of temperature and soil water potential and incorporated into the water retention curve and hydraulic conductivity functions with the intent of allowing the model to better capture some of the dynamic aspects of the strongly bound soil moisture that seems to require temperatures well beyond 150 C to fully evaporate. The model also can be configured to include film flow, although this phenomenon did not contribute much to the model’s overall performance. In general, the model simulates the laboratory-observed temperature dynamics quite well, but is less precise (but still good) at capturing the moisture dynamics. The model emulates the observed increase in soil moisture ahead of the drying front and the hiatus in the soil temperature rise during the strongly evaporative stage of drying. It also captures the observed rapid evaporation of soil moisture that occurs at relatively low temperatures (50-90 C), and can provide quite
accurate predictions of the total amount of soil moisture evaporated during the laboratory experiments. The model’s solution for water vapor density (and vapor pressure), which can exceed one standard atmosphere, cannot be experimentally verified, but they are supported by results from (earlier and very different) models developed for somewhat different purposes and for different porous media. Overall, this non-equilibrium model provides a much more physically realistic simulation over a previous equilibrium model developed for the same purpose. Current model performance strongly suggests that it is now ready for testing under field conditions.
1. Introduction

Since the development of the theory of Philip and de Vries (PdV model) almost 60 years ago [Philip and de Vries 1957; de Vries 1958] virtually all models of evaporation and condensation in unsaturated soils have assumed that soil water vapor at any particular depth into the soil is in equilibrium with the liquid soil water (or soil moisture) at the same depth. (Note: such soil evaporation models also assume thermal equilibrium, so that at any given depth the mineral soil, the soil moisture, and the soil air and water vapor within the pore space are also at the same temperature.) In essence, this local equilibrium assumption means that whenever the soil moisture changes phase it does so instantaneously. This assumption is quite apropos for its original application, which was to describe the coupled heat and moisture transport in soils (and soil evaporation in particular) under environmental forcings associated with the daily and seasonal variations in radiation, temperature, precipitation, etc. [e.g., Milly 1982; Novak 2010; Smits et al. 2011]. Under these conditions assuming local equilibrium is reasonable because the time required to achieve equilibrium after a change of phase is ‘instantaneous’ when compared to the time scale associated with solar and other normal environmental forcings. The great benefit to the equilibrium assumption is that for modeling purposes it is a significant simplification to the equations that describe heat and moisture flow in soils because it eliminates the need to include soil water vapor density, $\rho_v$, as an independent model variable. More formally, under equilibrium $\rho_v$ is directly equated to the equilibrium vapor density, which is a function only of local soil temperature and soil water content (or more specifically the soil water potential).

Subsequent to the development of the original PdV model the equilibrium assumption has also been incorporated into models of heat and moisture transport (evaporation and condensation) in soils and other porous media under more extreme forcings associated with high temperatures and heat fluxes. For example, it has been applied to (i) soils during
wildfires and prescribed burns [Aston and Gill 1976; Campbell et al. 1995; Durany et al. 2010; Massman 2012], (ii) drying of wood [Whitaker 1977; di Blasi 1997], (iii) drying and fracturing of concrete under high temperatures [Dayan 1982; Dal Pont et al. 2011], (iv) high temperature sand-water-steam systems [e.g., Udell 1983; Bridge et al. 2003], and (v) evaporation of wet porous thermal barriers under high heat fluxes [Costa et al. 2008].

Although the PdV model and the equilibrium assumption have certainly led to many insights into moisture and vapor transport and evaporation in porous media, it has, nonetheless, yielded somewhat disappointing simulations of the coupled soil moisture dynamics during fires [see Massman 2012 for further details and general modeling review]. Possibly the most interesting of these modeling “disappointments” is provided by soil/fire-heating model of Massman [2012], who found that as the soil moisture evaporated it just re-condensed and accumulated ahead of the dry zone, so that no water actually escaped the soil at all (which, to say the least, seems physically implausible)! He further traced the cause of this anomalous behavior to the inapplicability of the equilibrium evaporation assumption, which allowed the soil vapor gradient behind the drying front to become so small that the soil vapor could not escape (diffuse) out of the soil. Or, more fundamentally, the calculated vapor and its attendant gradient became largely meaningless because it is impossible for water vapor to be in equilibrium with liquid water if there is no liquid water. Of course, such extremely dry conditions are just about guaranteed during soil heating events like fires. Novak [2012] also recognized the inapplicability of the equilibrium assumption for very dry soils, but under more normal environmental forcing. On the other hand, even under normal (and much less extreme) soil moisture conditions both Smits et al. [2011] and Ouedraogo et al. [2013] suggest that non-equilibrium formulations of soil evaporation may actually improve model performance, which implies that the non-equilibrium assumption may really be a more appropriate description for soil evaporation and condensation than the equilibrium
assumption. The present study is intended to provide the first test of the non-equilibrium hypothesis during extreme conditions.

Specifically, the present study develops and evaluates a non-equilibrium (liquid-vapor phase change) model for simulating coupled heat, moisture, and water vapor transport during extreme heating events. It also assumes thermal equilibrium between the soil solids, liquid, and vapor. It uses a systems-theoretic approach [e.g., Gupta and Nearing 2014] focused more on physical processes than simply tuning model parameters, which here means that whatever model or parameter “tuning” does occur it is intended to keep the model numerically stable and as physically realistic as possible.

In addition, the present study (model) is a companion to Massman [2012]. It uses much of the same notation as the earlier study. But, unlike its predecessor, this study allows for the possibility of liquid water movement (i.e., it includes a hydraulic conductivity function for capillary and film flow). It also improves on and corrects (where possible and as noted in the text) the mathematical expressions used in the previous paper to parameterize the high temperature dependency of latent heat of vaporization, saturation vapor density, diffusivity of water vapor, soil thermal conductivity, water retention curve, etc. This is done in order to achieve the best representation of the physical properties of water (liquid and vapor) under high temperatures and pressures [see, e.g., Harvey and Friend 2004]. And lastly, in order to facilitate comparing the present model with the earlier companion model the present study displays all graphical results in a manner very similar to those of Massman’s [2012].

2. Model Development

The present model is one-dimensional (in the vertical) and is developed from three coupled partial differential equations and assumes that the soil liquid and vapor concentrations are not necessarily in local equilibrium during evaporation/condensation, but that they are in local thermal equilibrium during any phase change. Consequently the present model has
three simulation variables: soil temperature (≡ $T$ [C] or $T_K$ [K]); soil water potential (≡ $\psi$ [J kg$^{-1}$]) or $\psi_n \equiv$ normalized soil water potential [dimensionless], where $\psi_n = \psi/\psi_*$ and $\psi_* = -10^6$ [J kg$^{-1}$], which Campbell et al. [1995] identify as the water potential for oven dry soil); and vapor density (≡ $\rho_v$ [kg m$^{-3}$]). This current model employs a linearized Crank-Nicolson [C-N] finite difference scheme, whereas the preceding (companion) model [Massman 2012] used the Newton-Raphson method for solving the fully implicit finite difference equations. The present model further improves on its companion by including the possibility of soil water movement (hydraulic conductivity function driven by a gradient in soil moisture potential) and better parameterizations of thermophysical properties of water and water vapor to allow for large variations in the amount of soil water vapor, which Massman’s [2012] results suggest might approach or exceed one standard atmosphere and therefore could become the major component of the soil atmosphere during a heating event. This is quite unlike any other model of soil heat and moisture flow, which universally assume that dry air is the dominant component of the soil atmosphere and that water vapor is a relatively minor component. Finally, and also atypical of most other soil models, the model’s water retention curve and hydraulic function include a dynamic residual soil moisture content as a function of soil temperature and soil water potential.

2.1 Conservation Equations

The conservation of thermal energy is expressed as:

$$C_s \frac{dT}{dt} - \frac{\partial}{\partial z} \left[ \lambda_s \frac{\partial T}{\partial z} \right] + (\eta - \theta) \rho_a c_{pa} u_v l \frac{\partial T}{\partial z} = -L_v S_v$$

where $t$ [s] is time; $z$ [m] is soil depth (positive downward); $T$ is soil temperature in Celsius; $C_s = C_s(\theta, T)$ [J m$^{-3}$ K$^{-1}$] is the volumetric heat capacity of soil, a function of both soil temperature and soil volumetric water content $\theta$ [m$^3$m$^{-3}$]; $\lambda_s = \lambda_s(\theta, T, \rho_v)$ [W m$^{-1}$ K$^{-1}$]
is the thermal conductivity of the soil, a function of soil temperature, soil moisture, and soil vapor density; $\eta$ [m$^3$m$^{-3}$] is the total soil porosity from which it follows that $(\eta - \theta)$ is the soil's air filled porosity; $\rho_a = \rho_a(T_K, \rho_v)$ [kg m$^{-3}$] is the mass density of the soil air, a function of temperature and soil vapor density; $c_{pa} = c_{pa}(T_K, \rho_v)$ [J kg$^{-1}$ K$^{-1}$] is specific heat capacity of ambient air, also a function of temperature and vapor density; $u_{vl}$ [m s$^{-1}$] is the advective velocity induced by the change in volume associated with the rapid volitalization of soil moisture (detailed below); $L_v = L_v(T_K, \psi)$ [J kg$^{-1}$] is the latent heat of vaporization; and $S_v = S_v(T_K, \theta, \psi, \rho_v)$ [kg m$^{-3}$ s$^{-1}$] is the source term for water vapor.

The conservation of mass for liquid water is

$$\frac{\partial (\rho_w \theta)}{\partial t} - \frac{\partial}{\partial z} \left[ \rho_w K_n \frac{\partial \psi_n}{\partial z} + \rho_w K_H - \rho_w V_{\theta,surf} \right] = -S_v$$  \hspace{1cm} (2)

$\rho_w = \rho_w(T_K)$ [kg m$^{-3}$] is the density of liquid water; $K_n = K_n(T_K, \psi, \theta)$ [m$^2$ s$^{-1}$] is the hydraulic diffusivity; $K_H = K_H(T_K, \psi, \theta)$ [m s$^{-1}$] is the hydraulic conductivity; and $V_{\theta,surf} = V_{\theta,surf}(T_K, \theta)$ [m s$^{-1}$] is the velocity of liquid water associated with surface diffusion of water, which may be significant at high temperatures [e.g., Kapoor et al. 1989; Medveď and Černý 2011]. Note that switching variables from $\psi < 0$, to $\psi_n$ produces $\psi_n > 0$ and $K_n < 0$.

This last equation can be simplified to

$$\rho_w \frac{\partial \theta}{\partial t} - \rho_w \frac{\partial}{\partial z} \left[ K_n \frac{\partial \psi_n}{\partial z} + K_H - V_{\theta,surf} \right] = -S_v$$  \hspace{1cm} (3)

because $\frac{1}{\rho_w} \frac{d \rho_w}{dT}$ varies by only 4% between about 10 C to 100 C the derivatives $\frac{\partial \rho_w}{\partial t} \equiv \frac{d \rho_w}{dT} \frac{\partial T}{\partial t}$ and $\frac{\partial \rho_w}{\partial z} \equiv \frac{d \rho_w}{dT} \frac{\partial T}{\partial z}$ can be ignored. But the model does retain the temperature dependency $\rho_w = \rho_w(T_K)$, except as noted in the section below on volumetric specific heat capacity of soil, and it also specifically includes $d\rho_w/dT$ for other components of the model.
The conservation of mass for water vapor is

$$\frac{\partial (\eta - \theta) \rho_v}{\partial t} - \frac{\partial}{\partial z} \left[ D_{ve} \frac{\partial \rho_v}{\partial z} - (\eta - \theta) u_{el} \rho_v \right] = S_v$$  \hspace{1cm} (4)$$

where $D_{ve} = D_{ve}(T_K, \psi, \rho_v)$ [m$^2$ s$^{-1}$] is the (equivalent) molecular diffusivity associated with the diffusive transport of water vapor in the soil’s air-filled pore space.

The final model equations are expressed in terms of the model variables ($T, \psi_n, \rho_v$) and result from: (a) expanding the spatial derivative $\frac{\partial K_H}{\partial z}$ in terms of the spatial derivatives $\frac{\partial T}{\partial z}$ and $\frac{\partial \psi_n}{\partial z}$, (b) allowing for $\theta = \theta(\psi_n, T_K)$, (c) combining Equation (3) with (1) and Equation (4) with (3), and (d) simplifying Equation (4). These equations are

$$(C_s - L_v \rho_w D_{\theta T}) \frac{\partial T}{\partial t} - \frac{\partial}{\partial z} \left[ \lambda_n \frac{\partial T}{\partial z} \right] + \left[ (\eta - \theta) \rho_a c_p a u_{el} + L_v \rho_w \frac{\delta K_H}{\delta T_K} \right] \frac{\partial T}{\partial z} + L_v \rho_w \frac{\partial}{\partial z} \left[ K_m \frac{\partial T}{\partial z} \right]$$

$$- L_v \rho_w D_{\theta \psi} \frac{\partial \psi_n}{\partial t} + L_v \rho_w \frac{\partial}{\partial z} \left[ K_n^{\ast} \frac{\partial \psi_n}{\partial z} \right] + L_v \rho_w \frac{\partial}{\partial \psi_n} \left[ K_H \frac{\partial \psi_n}{\partial z} \right] = 0 \hspace{1cm} (5)$$

which is the conservation of energy;

$$\rho_w D_{\theta T} \frac{\partial T}{\partial t} - \rho_w \frac{\partial}{\partial z} \left[ K_m \frac{\partial T}{\partial z} \right] - \rho_w \left[ \frac{\delta K_H}{\delta T_K} \right] \frac{\partial T}{\partial z}$$

$$+ \rho_w D_{\theta \psi} \frac{\partial \psi_n}{\partial t} - \rho_w \frac{\partial}{\partial z} \left[ K_n^{\ast} \frac{\partial \psi_n}{\partial z} \right] - \rho_w \left[ D_{\psi \psi} \frac{\partial K_H}{\partial \theta} \right] \frac{\partial \psi_n}{\partial z}$$

$$+ (\eta - \theta) \frac{\partial \rho_v}{\partial t} - \frac{\partial}{\partial z} \left[ D_v \frac{\partial \rho_v}{\partial z} - (\eta - \theta) u_{el} \rho_v \right] = 0 \hspace{1cm} (6)$$
which is the conservation of soil moisture; and

\[-\rho_v D_{\theta T} \frac{\partial T}{\partial t} + (\eta - \theta) \frac{\partial \rho_v}{\partial t} - \frac{\partial}{\partial z} \left[ D_v \frac{\partial \rho_v}{\partial z} - (\eta - \theta) u_{el} \rho_v \right] - \rho_v D_{\theta \psi} \frac{\partial \psi_n}{\partial t} = S_v \]  

(7)

which is the conservation of mass for water vapor.

Apropos to these last three equations: (i) \( D_{\theta \psi} = \frac{\partial \theta}{\partial \psi_n} \) and \( D_{\theta T} = \frac{\partial \theta}{\partial T_K} \) are obtained from the water retention curve (WRC); (ii) \( \frac{\delta K_H}{\delta T_K} = \left[ \frac{\partial K_H}{\partial T_K} + \frac{\partial K_H}{\partial \theta} D_{\theta T} \right] \); (iii) \( K_m \text{ [m}^2\text{s}^{-1}\text{K}^{-1}] \) and \( K^*_n \text{ [m}^2\text{s}^{-1}] \) (which subsumes \( K_n \)) are related to \( V_{\theta,\text{surf}} \) and are defined in a later section; (iv) because \( \rho_v \ll \rho_w \) the term \( (\rho_w - \rho_v) \frac{\partial \theta}{\partial t} \) originally in Equation (6) has been approximated by \( \rho_w \frac{\partial \theta}{\partial t} \equiv \rho_w D_{\theta \psi} \frac{\partial \psi_n}{\partial t} + \rho_w D_{\theta T} \frac{\partial T}{\partial t} \); and (v) the total porosity \( \eta \) is assumed to be spatially uniform and temporally invariant.

2.2 Functional Parameterizations

2.2.1 Thermophysical Properties of Water, Vapor, and Moist Air

The algorithm for calculating water density, \( \rho_w(T_K) \), is Equation (2.6) of Wagner and Pruess [2002] and employed only within the temperature range \( 273.15 \text{ K} \leq T_K \leq 383.15 \text{ K} \) (\( \equiv T_{K,\text{max}} \)). At temperatures greater than \( T_{K,\text{max}} \), then \( \rho_w(T_K) = \rho_w(T_{K,\text{max}}) \). This approach yields a range for \( \rho_w(T_K) \) of \( 950 \text{ kg m}^{-3} < \rho_w(T_K) < 1000 \text{ kg m}^{-3} \), which represents a compromise between the fact that the density of (free saturated liquid) water continues to decrease with increasing temperatures [Yaws 1995] and the hypothetical possibility that in a bound state a mono-layer of liquid water \( \rho_w(T_K) \) may reach values as high as \( \approx 5000 \text{ kg m}^{-3} \) [Danielewicz-Ferchmin and Mickiewicz 1996]. \( d\rho_w/dT \) is computed from the analytical expression derived by differentiating the expression for \( \rho_w(T_K) \) and \( d\rho_w/dT = 0 \) for \( T_K > T_{K,\text{max}} \).

The enthalpy of vaporization of water, \( H_v = H_v(T_K, \psi) \text{ [J mol}^{-1}] \), is Equation (5) of
Somayajulu [1988] augmented by the soil moisture potential, $\psi$, [Massman 2012; Campbell et al. 1995] and is expressed as follows:

$$H_v = H_1 \left( \frac{T_{\text{crit}} - T_K}{T_K} \right) + H_2 \left( \frac{T_{\text{crit}} - T_K}{T_{\text{crit}}} \right)^{\frac{3}{2}} + H_3 \left( \frac{T_{\text{crit}} - T_K}{T_{\text{crit}}} \right)^{\frac{9}{4}} - M_w \psi$$  \hspace{1cm} (8)

where $H_1 = 13.405538 \text{ kJ mol}^{-1}$, $H_2 = 54.188028 \text{ kJ mol}^{-1}$, $H_3 = -58.822461 \text{ kJ mol}^{-1}$, $T_{\text{crit}} = 647.096 \text{ K}$ is the critical temperature for water, and $M_w = 0.01802 \text{ kg mol}^{-1}$ is the molar mass of water vapor. Note that $h_v = h_v(T_K)$ [$\text{J mol}^{-1}$] will denote the enthalpy of vaporization without the additional $-M_w \psi$ term, i.e., $h_v = h_v(T_K) = H_1 (T_{\text{crit}} - T_K)/T_K + H_2 [(T_{\text{crit}} - T_K)/T_{\text{crit}}]^\frac{3}{2} + H_3 [(T_{\text{crit}} - T_K)/T_{\text{crit}}]^\frac{9}{4}$. The present formulation differs from Massman [2012] because here $h_v(T_K \geq T_{\text{crit}}) = 0$; whereas Massman’s [and Campbell’s et al. 1995] equivalent $h_v$ was a linear approximation of the present $h_v$, which yielded $h_v(T_K \geq T_{\text{crit}}) \gg 0$. This distinction will become important when discussing the water vapor source term, $S_v$.

Note that because $L_v = H_v/M_w$ it also employs Equation (8).

The formulations for thermal conductivity of water vapor, $\lambda_v = \lambda_v(T_K)$ [$\text{W m}^{-1} \text{ K}^{-1}$], and liquid water, $\lambda_w = \lambda_w(T_K, \rho_w)$ [$\text{W m}^{-1} \text{ K}^{-1}$], are taken from Huber et al. [2012]. For water vapor their Equation (4) is used and for liquid water the product of their Equations (4) and (5) is used. The formulations for viscosity of water vapor and liquid water are taken from Huber et al. [2009] and are similar algorithmically to thermal conductivity. For water vapor, $\mu_v = \mu_v(T_K)$ [$\text{kg m}^{-1} \text{ s}^{-1} \equiv \text{Pa s}$], their Equation (11) is used and for liquid water, $\mu_w = \mu_w(T_K, \rho_w)$ [$\text{Pa s}$], their Equation (36) is used. For liquid water both these formulations include a dependence on the density of water. Consequently, once soil temperature exceeds $T_{K,\text{max}}$ both $\lambda_w$ and $\mu_w$ are assigned a fixed value determined at $T_{K,\text{max}}$. On the other hand, $\lambda_v$ and $\mu_v$ increase continually with increasing temperatures.

The formulation for the thermal conductivity of dry air, $\lambda_d = \lambda_d(T_K)$ [$\text{W m}^{-1} \text{ K}^{-1}$], is
Equation (5a) of Kadoya et al. [1985] and for the viscosity of dry air, $\mu_d = \mu_d(T_K)$ [Pa s], Equation (3a) of Kadoya et al. [1985] is used. The model of the thermal conductivity of soil atmosphere, $\lambda_a = \lambda_a(\lambda_v, \lambda_d, \mu_v, \mu_d)$ [W m$^{-1}$ K$^{-1}$], is a non-linear expression given by Equation (28) of Tsilingiris [2008]. The relative weights used in this formulation are determined using the mixing ratios for water vapor ($\chi_v$ [dimensionless]) and dry air ($\chi_d = 1 - \chi_v$): where $\chi_v = e_v/(P_d + e_v)$, $e_v$ [Pa] is the vapor pressure, and $P_d$ [Pa] is the dry air pressure. Here $P_d$ will be held constant and equal to the ambient atmospheric pressure, $P_{atmos}$ [= 92 kPa], during the laboratory experiments [see Massman 2012; Campbell et al. 1995]. The vapor pressure, $e_v$, is obtained from $\rho_v$ and $T_K$ using the ideal gas law.

The volumetric specific heat for soil air, $\rho_a c_{pa}$ [J m$^{-3}$ K$^{-1}$], is estimated for the soil atmosphere from $\rho_a c_{pa} = c_{pv} \rho_v + c_{pd} \rho_d$; where $\rho_d = M_d P_{atmos}/(RT_K)$ [kg m$^{-3}$] is the dry air density, $R = 8.314$ Jmol$^{-1}$K$^{-1}$ is the universal gas constant, $M_d = 0.02896$ kg mol$^{-1}$ is the molar mass of dry air, and the isobaric specific heats for water vapor, $c_{pv}$ [J kg$^{-1}$ K$^{-1}$], and dry air, $c_{pd}$ [J kg$^{-1}$ K$^{-1}$], use Equation (6) of Bücker et al. [2003].

The saturation vapor pressure, $e_{v,sat} = e_{v,sat}(T_K)$ [Pa], and its derivative, $de_{v,sat}/dT$ [Pa K$^{-1}$], are modeled using Equations (2.5) and (2.5a) of Wagner and Pruess [2002]. The saturation vapor density, $\rho_{v,sat}$ [kg m$^{-3}$], is modeled using Equation (2.7) of Wagner and Pruess [2002]. Following Massman [2012], these saturation curves are restricted to temperatures below that temperature, $T_{K,sat}$, at which $e_{v,sat} = P_{atmos}$. For the present case $T_{K,sat} = 370.44$ K was determined from the saturation temperature equation or “the Backward Equation”, Equation (31) of IAPWS [2007]. For $T_K \geq T_{K,sat}$ the saturation quantities $e_{v,sat}$ and $de_{v,sat}/dT$ remain fixed at their values $T_{K,sat}$, but $\rho_{v,sat}$ is allowed to decrease with increasing temperatures, i.e., $\rho_{v,sat} = \rho_{v,sat}(T_{K,sat})[T_{K,sat}/T_K][P_{atmos}/P_{ST}]$, in accordance with Table 13.2 (page 497) of Wagner and Pruess [2002], where $P_{ST} = 101325$ Pa is the standard pressure. The present treatment of $\rho_{v,sat}$ is different from Massman [2012], who assumed
that $\rho_{v,\text{sat}}(T_K \geq T_{K,\text{sat}}) = \rho_{v,\text{sat}}(T_{K,\text{sat}})$.

Embedded in the hydraulic conductivities $[K_H$ and $K_n$ of Equation (2)] are the surface tension of water, $\sigma_w(T_K)$ [N m$^{-1}$], and the static dielectric constant (or the relative permittivity) of water, $\epsilon_w(T_K)$ [dimensionless]. These physical properties of water are integral to Zhang’s [2011] model for the hydraulic conductivity associated with film flow, which is incorporated into the present model and detailed in a later section. The surface tension of water is modeled following Equation (1) of Vargaftik et al. [1983] and $\epsilon_w(T_K)$ is taken from Equation (36) of Fernández et al. [1997].

2.2.2 Functions Related to Water Vapor: $D_{ve}$, $u_{vl}$, $S_v$

$D_{ve}$ is modeled as:

$$D_{ve} = \frac{\tau(\eta - \theta)S_F\mathcal{E}D_v}{1 + e_v/P_{atmos}}$$

where $\tau$ [m m$^{-1}$] is the tortuosity of soil with $\tau = 0.66[(\eta - \theta)/\eta]^3$ after Moldrup et al. [1997], $\mathcal{E}$ [dimensionless] is the vapor flow enhancement factor and is discussed in Massman [2012], $D_v/(1 + e_v/P_{atmos})$ [m$^2$ s$^{-1}$] is the molecular diffusivity of water vapor into the soil atmosphere, which will be taken as a mixture of both dry air and (potentially large amounts of) water vapor, and $S_F$ [dimensionless] is Stefan correction or mass flow factor. Externalizing the $1/(1 + e_v/P_{atmos})$ term of the vapor diffusivity and combining it with $S_F$ allows for the following approximation for $S_F/(1 + e_v/P_{atmos}) = 1/(1 - e_v^2/P_{atmos}^2) \approx 1 + e_v/P_{atmos}$; where the correct form for $S_F$ is $1/(1 - e_v/P_{atmos})$. The reason for this approximation is to avoid dividing by 0 as $e_v \rightarrow P_{atmos}$. Massman [2012] and Campbell et al. [1995] avoided this issue by limiting $S_F$ to a maximum value of 10/3. This newer approximation for $S_F/(1 + e_v/P_{atmos})$ is an improvement over their approach. It is relatively slower to enhance the vapor transport by diffusion than the original approach, but this is only of any real significance when $S_F \geq 1$. On the other hand because the linear form is not limited to any preset maximum value it compensates for these underestimations when $S_F > 10/3$. 

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Mindful of the externalization of \(1/(1 + e_v/P_{atmos})\), \(D_v\) is estimated from the diffusivity of water vapor in dry air, \(D_{vd} = D_{vd}(T_K)\) [m² s⁻¹] and the self-diffusivity of water vapor in water vapor, \(D_{vv} = D_{vv}(T_K)\) [m² s⁻¹], where

\[
D_{vd} = D_{vdST} \left( \frac{P_{ST}}{P_{atmos}} \right) \left( \frac{T_K}{T_{ST}} \right)^{\alpha_{vd}}
\]

and

\[
D_{vv} = D_{vvST} \left( \frac{P_{ST}}{P_{atmos}} \right) \left( \frac{T_K}{T_{ST}} \right)^{\alpha_{vv}}
\]

and \(T_{ST} = 273.15\) K is the standard temperature, \(D_{vdST} = 2.12 \times 10^{-5}\) m² s⁻¹, \(\alpha_{vd} = 7/4\), \(D_{vvST} = 1.39 \times 10^{-5}\) m² s⁻¹, and \(\alpha_{vv} = 9/4\). The parameters \(D_{vvST}\) and \(\alpha_{vv}\) relate to the self-diffusion of water vapor and their numerical values were determined from a synthesis of results from Hellmann et al. [2009] and Yoshida et al. [2006, 2007]. The uncertainty associated with this value for \(D_{vvST}\) is at least ±15% and possibly more, e.g., Miles et al. [2012]. Blanc’s Law [Marrero and Mason 1972] combines \(D_{vd}\) and \(D_{vv}\) to yield the following expressions for \(D_v = D_v(T_K, \rho_v)\):

\[
\frac{1}{D_v(T_K, \rho_v)} = \frac{1 - \chi_v}{D_{vd}} + \frac{\chi_v}{D_{vv}} \quad \text{or} \quad D_v(T_K, \rho_v) = \frac{D_{vd}D_{vv}}{(1 - \chi_v)D_{vv} + \chi_vD_{vd}}
\]

The present model for the advective velocity associated with the volatilization of water, \(u_{vl}\), is taken from Ki et al. [2005] and is non-equilibrium equivalent to that used by Massman [2012] in his equilibrium model. Here

\[
\frac{\partial u_{vl}}{\partial z} = \frac{S_v}{(\eta - \theta)\rho_v}
\]

(9)
where the basic assumptions are that both liquid water and vapor are Newtonian fluids and that only incompressible effects are being modeled. In essence Equation (9) assumes that the vaporization of soil moisture acts as a steady-state (and rapidly expanding or “exploding”) volume source term, which yields a 1-D advective velocity associated with volatilization of liquid water. For an equilibrium model of soil moisture evaporation that does not include water movement (i.e., $K_H \equiv 0$, $K_n \equiv 0$, and $V_{\theta, surf} \equiv 0$), then $S_v \equiv -\rho_w \partial \theta / \partial t$ (from Equation (2) above), which demonstrates the connection between present model of $u_{vl}$ with that used by Massman’s [2012]. But unlike Massman [2012], the present model does not require any numerical adjustments to Equation (9) in order to maintain numerical stability.

The functional parameterization of $S_v$ follows from the non-equilibrium assumption, i.e.,

$$S_v \propto (\rho_{ve} - \rho_v),$$

where $\rho_{ve} = a_w \rho_{v, sat}(T_K)$ [kg m$^{-3}$] is the equilibrium vapor density and $a_w = e^{M_w \psi} / \psi$ [dimensionless] is the water activity, modeled here with the Kelvin Equation. The difficult part is how to construct the proportionality coefficient. Nevertheless, there are at least two ways to go about this: (a) largely empirically [e.g., Smits et al. 2011 and related approaches referenced therein], or (b) assume that $S_v = A_{wa} J_v$ [e.g., Skopp 1985 or Novak 2012], where $A_{wa}$ [m$^{-1}$] is the volume-normalized soil water-air interfacial surface area and $J_v$ [kg m$^{-2}$ s$^{-1}$] is the flux to/from that interfacial surface. This second approach allows a more physically-based parameterization of the flux, viz., $J_v = R_v (\rho_{ve} - \rho_v)$, where $R_v$ [ms$^{-1}$] is the interfacial surface transfer coefficient. For example, Novak [2012] proposed that the flux be driven be diffusion, so that $R_v = D_v / r_{ep}$, where $r_{ep}$ [m] is the equivalent pore radius and $D_v$ is the diffusivity of water vapor in soil air. After a bit of algebra and some simple geometrically-based assumptions concerning the relationships between $r_{ep}$, a spherical pore volume, and $A_{wa}$, one arrives at ("the Novak") model of the source term:
\[ S_v^{[N]} = S_v^{[N]} A_{wa}^2 D_v (\rho_{ve} - \rho_v) \]

where \( S_v^{[N]} \) [dimensionless] is an adjustable model parameter.

But there is another way to model the vapor flux, \( J_v \), which is also used in the present study. This second approach is based on the Hertz-Knudsen Equation, which has its origins in the kinetic theory of gases and describes the net flux of a gas that is simultaneously condensing on and evaporating from a surface. A general expression for the Hertz-Knudsen flux is

\[ J_v = \sqrt{\frac{RT_K}{M_w}} \left( \frac{\rho_{ve}}{\rho_v} - K_c \right) \]

where \( K_e \) [dimensionless] is the mass accommodation (or evaporation) coefficient and \( K_c = K_c(T_K, \psi_n) \) [dimensionless] is the thermal accommodation (or condensation) coefficient. For the present purposes \( K_e \equiv 1 \) can be assumed. This model of \( J_v \) yields the following model for \( S_v \):

\[ S_v^{[M]} = S_v^{[M]} A_{wa} \sqrt{\frac{RT_K}{M_w}} (\rho_{ve} - K_c \rho_v) \] (10)

where \( S_v^{[M]} \) [dimensionless] is an adjustable model parameter, to be determined by “tuning” it as necessary to ensure model stability. This model for \( S_v^{[M]} \) is now more or less complete, but the model for \( S_v^{[N]} \) is neither quite complete nor precisely comparable to \( S_v^{[M]} \). This is now remedied by introducing \( K_c \) into \( S_v^{[N]} \) and subsuming a factor of \( A_{wa} \) into \( S_v^{[N]} \), yielding:

\[ S_v^{[N]} = S_v^{[N]} A_{wa} D_v (\rho_{ve} - K_c \rho_v) \] (11)

where \( S_v^{[N]} \) [m\(^{-1}\)] now has physical dimensions, but otherwise remains an adjustable parameter that will be scaled such that \( S_v^{[N]} \approx O(S_v^{[M]}) \). In this form these two models for \( S_v \)
can be used to test the sensitivity of the model’s solution to different temperature forcing, because $S_v^{[M]} \propto \sqrt{T_K}$, whereas $S_v^{[N]} \propto T_K^\alpha$, where $\alpha \geq 2$.

Concluding the development of $S_v$ requires models of $K_c$ and $A_{wa} = A_{wa}(\theta)$. $K_c$ is parameterized as

$$K_c(T_K, \psi_n) = e^{E_{av} - M_w \psi/RT} \left( \frac{1}{T_K} - \frac{1}{T_{K,in}} \right)$$

where $T_{K,in} [K]$ is the initial temperature of the laboratory experiments and because $T_K \geq T_{K,in}$ throughout the experiment $K_c \leq 1$ throughout the experiment and $E_{av} - M_w \psi [J \text{ mol}^{-1}]$ is an empirically determined surface condensation/evaporation activation energy. (Note: the enthalpy of vaporization, $h_v(T_K)$, is a logical choice for $E_{av}$. Nevertheless, model performance was significantly enhanced by simply assigning a constant value for $E_{av} \approx 30 - 40 \text{ kJ mol}^{-1}$ rather than using $h_v$.) The present formulation ensures that $\partial K_c/\partial T_K < 0$, in accordance with experimental and theoretical studies [Tsuruta and Nagayama 2004; Kon et al. 2014]. Mathematically this present formulation of $K_c$ largely eliminates model instabilities by suppressing condensation relative to evaporation throughout the experiment and will be discussed in greater detail in a later section.

$A_{wa}$ is parameterized as a parabolic function to simulate the conceptual model of $A_{wa}$ proposed by Constanza-Robinson and Brusseau [2002: see their Figure (1b)]:

$$A_{wa}(\theta) = S_w(1 - S_w)^{a_1} + a_2[S_w(1 - S_w)]^{a_3}$$

where $S_w = \theta/\eta$ is the soil water saturation and $a_1 = 40$, $a_2 = 0.003$, and $a_3 = 1/8$. This particular functional form ensures that $A_{wa} = 0$ when the soil is completely dry, $\theta = 0$, and when fully saturated, $S_w = 1$. This particular parameter value for $a_1$ was chosen so that
the maximum value of $A_{wa}$ occurs at $S_w = 0.025 (=1/a_1)$ in accordance with the model of Constanza-Robinson and Brusseau [2002].

### 2.2.3 Thermal Transport Properties: $C_s$, $\lambda_s$

The model for $C_s(T, \theta)$ is taken from Massman [2012]:

$$C_s(T, \theta) = c_s(T)\rho_b + C_w(T)\theta,$$

where $\rho_b$ [kg m$^{-3}$] is the soil bulk density; $c_s(T) = c_{s0} + c_{s1}T$ [J kg$^{-1}$ K$^{-1}$] is the specific heat capacity of soil; and $C_w(T) = C_{w0} + C_{w1}T + C_{w2}T^2$ [J m$^{-3}$ K$^{-1}$] is the volumetric heat capacity of water; and the parameterization constants $c_{s0}, c_{s1}, C_{w0}, C_{w1}, C_{w2}$ are given by Massman [2012]. Note that the present $C_s(T, \theta)$ results from approximating $C_w(T) \equiv c_{pw}(T)\rho_w(T_K)$ by $c_{pw}(T)\rho_w(T_{ST})$, where $c_{pw}(T)$ [J kg$^{-1}$ K$^{-1}$] is the isobaric specific heat capacity of water, and $\rho_w(T_{ST}) = 1000$ kg m$^{-3}$. This substitution for $\rho_w(T_K)$ is only made for $C_w(T)$.

The present formulation for isobaric heat capacity of water, $c_{pw}(T)$, was developed from Yaws [1995] and confirmed by comparing to Wagner and Pruess [2002]. In general $c_{pw}(T)$ is also a function of pressure [e.g., Wagner and Pruess 2002], but this dependency can be ignored for the present purposes. Other parameterization of $c_{pw}(T)$ [i.e., Sato 1990; Jovanović et al. 2009; Kozlowski 2012] were also examined, but proved unsatisfactory. Finally, Kozlowski [2012] reports numerical values for the dry soil parameters $c_{s0}$ and $c_{s1}$ that are similar to those discussed in Massman [2012] and used with the present model.

The model of soil thermal conductivity, $\lambda_s$, is the sum of two terms. The first, $\lambda_s^{[1]}(T_K, \rho_v)$, is taken principally from Campbell et al. [1994] and the second, $\lambda_s^{[2]}(T_K)$, is taken from Bauer [1993]. This second term incorporates the effects of high-temperature thermal (infrared) radiant energy transfer within the soil pore space, which may be significant for certain soils and high enough temperatures [e.g., Durany et al. 2010]. $\lambda_s^{[1]}$ is summarized first and repeated here to emphasize the difference between the present model’s functional parameterizations and those used in Massman [2012]. $\lambda_s^{[1]}$ is modeled as:
\[
\lambda_{s}^{[1]}(\theta, T_K, \rho_v) = \frac{k_w \theta \lambda_w(T_K, \rho_w) + k_a [\eta - \theta] \lambda_a^{*}(\theta, T_K, \rho_v) + k_m [1 - \eta] \lambda_m}{k_w \theta + k_a [\eta - \theta] + k_m [1 - \eta]}
\]  
(12)

where \(\lambda_a^{*}(\theta, T_K, \rho_v) = \lambda_a(T_K) + \lambda_a^{*}(\theta, T_K, \rho_v)\) is the apparent thermal conductivity of the soil air and is the sum of the thermal conductivity of moist soil air, \(\lambda_a\), and \(\lambda_a^{*}\), which incorporates the effects of latent heat transfer; \(\lambda_m\) is the thermal conductivity of the mineral component of the soil, which is assumed to be independent of temperature and soil moisture; and \(k_w\), \(k_a\), and \(k_m\) \([\text{dimensionless}]\) are generalized formulations of the de Vries weighting factors \([\text{de Vries 1963}]\). \(\lambda_a^{*}\) as proportional to the product of the enthalpy of vaporization \((h_v)\), the vapor diffusivity \((D_v)\), the Stefan factor \((S_F)\), the slope of the saturation vapor pressure \((d e_{v, sat}/dT)\), and the parameter \(f_w(\theta, T)\) \((\text{Campbell et al. 1995 \[1995\] also used in the diffusivity enhancement factor, } E)\). For present model \(\lambda_a^{*}\) is

\[
\lambda_a^{*}(\theta, T_K, \rho_v) = \frac{\hat{\rho}_{ST} h_v f_w S_F D_v [d e_{v, sat}/dT]}{P_{\text{atmos}}}
\]  
(13)

where \(\hat{\rho}_{ST} = 44.65 \text{ mol m}^{-3}\) is the molar density of the standard atmosphere. Equations (12) and (13) are the same as those used in \(\text{Massman} [2012]\), but numerically they yield quite results due to the different formulations for \(h_v, S_F, D_v,\) and \(e_{v, sat} = e_{v, sat}(T_K)\). Otherwise the \(\text{de Vries} [1963]\) shape factors, the parameter \(f_w\), and all related parameters are the same as in \(\text{Massman} [2012].\)

\(\lambda_{s}^{[2]}\) is modeled as:

\[
\lambda_{s}^{[2]}(\theta, T_K) = 3.8 \sigma N^2 R_{p} T_K^3
\]  
(14)
where $\sigma = 5.670 \times 10^8$ W m\(^{-2}\) K\(^{-4}\) is the Boltzmann constant; $N = N(\theta) = 1 + \theta/(3\eta)$ is the medium’s [dimensionless] index of refraction; $R_p$ [m] is the soil’s pore space volumetric radius; and the factor of 3.8 subsumes a numerical factor of 4, a [dimensionless] pore shape factor (= 1 for spherical particles), and the [dimensionless] emissivity of the medium $\approx 0.95$ (by assumption). In general $R_p$ is considered to be a model parameter and it will be varied to evaluate the model’s sensitivity and performance relative to $\lambda_s^{[2]}$. But unless otherwise indicated, $R_p = 10^{-3}$ m is the nominal or default value.

2.2.4 Water Retention Curve

In general a WRC is a functional relationship between soil moisture and soil moisture potential and temperature, i.e., $\theta = \theta(\psi, T)$, although the temperature dependency is often ignored and was of little consequence to Massman’s [2012] model. The three WRCs tested in the present study have been adapted to include a residual soil moisture, $\theta_r = \theta_r(\psi, T)$ [m\(^3\)m\(^{-3}\)], which is an atypical parameterization for both $\theta_r$ and the soil moisture’s temperature dependency. Under more normal soil environmental conditions $\theta_r$ is assumed to be bound so securely to soil mineral surfaces that it is normally taken as a fixed constant. For the present purposes the principal WRC is adapted from Massman [2012] and is

$$
\theta(\psi_n, T_K) = -\frac{\theta_l}{\alpha_l} \ln (\psi_n) + \left[ \theta_h - \theta_r(\psi_n, T_K) \right] \left[ 1 + (\alpha_h \psi_n)^4 \right]^{-\frac{p}{p}} + \theta_r(\psi_n, T_K) \quad (15)
$$

Where

$$
\theta_r(\psi_n, T_K) = \theta_r e^{\frac{b_1 E_{av}(1-b_2 \psi_n)}{R_p \left( \frac{1}{T_K} - \frac{1}{T_{K,in}} \right)}} \quad (16)
$$

and $\theta_l$ [m\(^3\) m\(^{-3}\)] is the extrapolated value of the water content when $\psi = \psi_l = -1$ J kg\(^{-1}\); $\alpha_l = \ln (\psi_s/\psi_l) = 13.8155106$; and $\theta_h$ [m\(^3\) m\(^{-3}\)], $\alpha_h$ [dimensionless] and $p$ [dimensionless]
are parameters obtained from *Campbell and Shiozawa* [1992]; $\theta_r^*$ [$m^3\ m^{-3}$] is a constant soil-specific parameter, such that $\theta_r^* \leq 0.03$ is to be expected, and $b_1$ [dimensionless] and $b_2$ [dimensionless] are adjustable parameters, which are expected to satisfy $b_1 > 0$ and $0 \leq b_2 < 1$. Note: further discussion concerning the original version of Equation (15) can be found in *Massman* [2012].

There is a simple and physically intuitive argument for this particular parameterization of $\theta_r(\psi_n, T_K)$ in Equation (16). First, under more normal soil environmental conditions, i.e., $T_K \approx T_{K,\text{in}}$ and at least $\psi_n < 1$ (if not $\psi_n \ll 1$), then it is reasonable to expect that $\theta_r \approx \theta_r^*$ and nearly constant throughout (what might be expected to be relatively small variations in) those conditions. But as the temperature increases, it is also reasonable to assume that the increasing amounts of thermal energy will begin to overcome the forces holding the bound water to the soil mineral surfaces and that $\theta_r$ will decrease. Mathematically, then one might therefore expect that $\partial \theta_r/\partial T_K < 0$. *Massman* [2012: see his discussion of $\psi_T$] made similar arguments when he included the temperature dependency in his version of the same basic WRC. Consequently, the Equations (15) and (16) above offer a different approach to including temperature effects on the WRC that maintains the temperature dependent properties of WRCs outlined by *Massman* [2012]. Second, as the temperature increases and the soil moisture (including $\theta_r$) begins to decrease, the soil moisture potential $\psi_n$ will begin to increase (or $\psi < 0$ will decrease in absolute terms while increasing in magnitude), which in turn (it is hypothesized) will tend to strengthen the forces holding the bound water. Therefore, one might expect that as the soil dries out $\partial \theta_r/\partial \psi_n > 0$, which will oppose, but not dominate the temperature effects, i.e., $b_2 < 1$. Equation (16) is designed to capture these two opposing influences, assuming of course that $T_K > T_{K,\text{in}}$. But, it is itself not intended to be a fully physically-based dynamical theory of the residual soil moisture. Such a theory is beyond the intent of the present study. The sole intent here is to test and evaluate whether a
dynamical $\theta_r$ can improve the model’s performance. And in so far as it may succeed at doing so, it will also indicate the value and need for a more detailed physically-based dynamical model of $\theta_r$.

The present study also includes similar adaptations to two other WRCs so as to test the model’s sensitivity to different WRCs. These WRCs, which will not be shown here, are taken from Groenevelt and Grant [2004] and Fredlund and Xing [1994].

2.2.5 Functions Related to Liquid Water Transport: $K_n$, $K_H$, $V_{\theta,surf}$

The hydraulic conductivity functions, $K_n(T_K, \psi_n, \theta)$ and $K_H(T_K, \psi_n, \theta)$, are given as follows:

$$K_n = \frac{K_I K_R \rho_w \psi^*}{\mu_w} \quad \text{and} \quad K_H = \frac{K_I K_R \rho_w g}{\mu_w}$$

where $g = 9.81 \text{ m s}^{-2}$ is the acceleration due to gravity; $K_I \text{ [m}^2\text{]}$ is the intrinsic permeability of the soil (a constant for any given soil); and $K_R = K_R(\theta, \theta_r, \psi_n, T_K)$ [dimensionless] is the hydraulic conductivity function (HCF), which for the present study is taken as the sum of $K_{cap}^R$ (associated with capillary flow) and $K_{film}^R$ (associated with film flow). The model for intrinsic permeability, which is taken from Bear [1972], is $K_I = (6.17 \times 10^{-4})d_g^2$; where $d_g \text{ [m]}$ is the mean or ‘effective’ soil particle diameter. For the soils used in the present work [Campbell et al. 1995; Massman 2012] $d_g$ was estimated from Shiozawa and Campbell [1991] and Campbell and Shiozawa [1992] or simply assigned a reasonable value if no other information was available.

For present study five difference parameterizations for $K_{cap}^R$ were tested. Two were from Grant et al. [2010], i.e., their Equation (18) [Burdine] and Equation (19) [Mualem]; the Van Genuchten and Nielson [1985] model, their Equation(22) with the mathematical constraints imposed as suggested by Assouline and Or [2013]; the Brooks and Corey [1964] model; and
Equation (18) of Assouline [2001]. The reason for testing several models of the HCF is to determine how different formulations for the HCF might impact the model’s performance when comparing to the laboratory observations. The following HCF is Assouline’s [2001] model, which is a relatively simple formulation for the HCF and serves as the reference HCF for the model simulations.

\[
K_{R}^{\text{cap}}(\theta, \theta_r) = \left(1 - \left[1 - \left(\frac{\theta - \theta_r(\psi_n, T_K)}{\eta} \right)^{\frac{1}{m}}\right]^n\right) \quad (17)
\]

where for the present application \(0 < m < 1\), and \(n > 1\).

\[
K_{R}^{\text{film}}(T_K, \psi_n) = \frac{R_w(T_K)}{6.17 \times 10^{-4}} \left[\frac{2\sigma_w}{2\sigma_w - \rho_w d_y \psi_n \psi_n}\right]^{\frac{1}{2}} \quad (18)
\]

and

\[
R_w(T_K) = \sqrt{2} \pi^2 (1 - \eta) F_w \left[\left(\frac{\epsilon_0 \epsilon_w}{2\sigma_w d_y}\right) \left(\frac{k_B T_K}{z a}\right)\right]^3 \quad (19)
\]

where (as above) \(\sigma_w(T_K)\) is the surface tension of water and \(\epsilon_w(T_K)\) is the static dielectric constant or relative permittivity of water; \(\epsilon_0 = 8.85 \times 10^{-12} \text{ C}^2 \text{ J}^{-1} \text{ m}^{-1}\) is the permittivity of free space; \(k_B = 1.308568 \times 10^{-23} \text{ J K}^{-1}\) is the Boltzmann constant; \(a = 1.6021773 \times 10^{-19} \text{ C}\) is the electron charge; \(z\) [dimensionless] is the ion charge, for which \(z = 1\) can be assumed; and \(F_w\) [dimensionless] is a soil-specific parameter, for which Zhang [2011] found that (roughly) \(10 < F_w < 10^4\).
The term $\rho_w V_{\theta, surf}$ in Equation (2) represents the soil moisture movement caused by water molecules “hopping” or “skipping” along the surface of the water films due to a temperature gradient [e.g., Medved and Černý 2011]. The present model for $V_{\theta, surf}$ is adapted from the model of Gawin et al. [1999] and is given as:

$$V_{\theta, surf} = -D_{\theta s} \frac{\partial \theta}{\partial z} = -D_{\theta s} D_{\theta \psi} \frac{\partial \psi_n}{\partial z} - D_{\theta s} D_{\theta T} \frac{\partial T}{\partial z}$$  \hspace{1cm} (20)

where $D_{\theta s} = D_{\theta s}(T_K, \theta)$ [m$^2$s$^{-1}$] is the surface diffusivity and is parameterized as:

$$D_{\theta s} = D_{\theta s0} \exp \left[ -2 \left( \frac{\theta}{\theta_b} \right)^\beta \left( \frac{T_{ST}}{T_K} \right) \right]$$  \hspace{1cm} (21)

with $D_{\theta s0} \approx 10^{-10}$ m$^2$s$^{-1}$; $\theta_b \approx 0.02$; and $\beta = 1/4$ when $\theta \geq \theta_b$ or otherwise $\beta = 1$ when $\theta < \theta_b$.

By expressing $V_{\theta, surf}$ in terms of the gradient of the “normalized” soil moisture potential, $\psi_n$, in Equation (20), $K^*_n$ and $K_m$, used in Equations (5) and (6), can be identified as:

$$K^*_n = K_n + D_{\theta s} D_{\theta \psi} \quad \text{and} \quad K_m = D_{\theta s} D_{\theta T}; \quad \text{where} \quad D_{\theta s} D_{\theta \psi} \quad \text{will be defined as} \quad K_{n, surf}; \quad \text{where} \quad K_{n, surf} \leq 0 \ \forall \ T_K \ \text{and} \ \theta.$$

### 3. Numerical Implementation

The numerical model as outlined above and detailed in this section is coded as MATLAB (The MathWorks Inc., Natick, MA, Version R2013b) script files.

#### 3.1 Crank-Nicolson Method

The linearized Crank-Nicolson method is used to solve Equations (5), (6), and (7). For Equation (5) this yields the following (canonical) linear finite difference equation:

$$-A_{TTi}^{ij} T_{ti-1}^{j+1} + \left[ 1 + B_{TTi}^{ij} \right] T_{ti}^{j+1} - C_{TTi}^{ij} T_{ti+1}^{j+1} + A_{T\psi_i}^{ij} \psi_{mi}^{j+1} + \left[ \Gamma_{T\psi_i}^{ij} - B_{T\psi_i}^{ij} \right] \psi_{ni}^{j+1} = C_{T\psi_i}^{ij} \psi_{ni}^{j+1}.$$
\[ A_{T_{TTi}}^j T_{i-1}^j + \left[ 1 - B_{T_{TTi}}^j \right] T_i^j + C_{T_{TTi}}^j T_{i+1}^j - A_{T_{T\psi{i}}}^j \psi_{ni-1}^j - \left[ \Gamma_{T_{T\psi{i}}}^j + B_{T_{T\psi{i}}}^j \right] \psi_{ni}^j - C_{T_{T\psi{i}}}^j \psi_{ni+1}^j \]  \tag{22}

where \( j \) and \( j + 1 \) are consecutive time indices, \( i - 1, i, \) and \( i + 1 \) are contiguous spatial indices, and \( A_{T_{TTi}}^j, B_{T_{TTi}}^j, C_{T_{TTi}}^j, A_{T_{T\psi{i}}}^j, B_{T_{T\psi{i}}}^j, C_{T_{T\psi{i}}}^j, \) and \( \Gamma_{T_{T\psi{i}}}^j \) are the linearized C-N coefficients, which will not be explicitly listed here, but they do largely follow conventions and notation similar to Massman [2012]. Although containing more terms than Equation (22), the finite difference equation corresponding to Equation (6) is very similar. But to linearize Equation (7), the Crank-Nicolson scheme requires linearizing the source term, \( S_v(T, \theta, \psi, \rho_v) \). This is done with a first order Taylor series expansion of the C-N term \( S_v^{j+1} \) as follows:

\[ S_v^{j+1} = S_v^j + \left( \frac{\delta S_v}{\delta T} \right)_i^j (T_{i+1}^j - T_i^j) + \left( \frac{\delta S_v}{\delta \psi_n} \right)_i^j (\psi_{ni}^{j+1} - \psi_{ni}^j) + \left( \frac{\partial S_v}{\partial \rho_v} \right)_i^j (\rho_{vi}^{j+1} - \rho_{vi}^j) \]

where \( \frac{\delta S_v}{\delta T} = D_{\theta T} \frac{\partial S_v}{\partial \theta} + \frac{\partial S_v}{\partial T} \) and \( \frac{\delta S_v}{\delta \psi_n} = D_{\psi_n} \frac{\partial S_v}{\partial \theta} + \frac{\partial S_v}{\partial \psi_n} \), which in turn yields the following linearized finite difference equation for Equation (7):

\[ - \left[ \frac{\Delta t}{2(\eta - \theta)_i^j} \left( \frac{\delta S_v}{\delta T} \right)_i^j \right] T_{i+1}^j \left[ 1 + B_{\rho \psi{i}}^j \frac{\Delta t}{2(\eta - \theta)_i^j} \left( \frac{\partial S_v}{\partial \rho_v} \right)_i^j \right] - \rho_{vi}^{j+1} - C_{\rho \psi{i}}^j \rho_{vi}^{j+1} = \]

\[ - \left[ \frac{\Delta t}{2(\eta - \theta)_i^j} \left( \frac{\delta S_v}{\delta T} \right)_i^j \right] T_i^j \left[ B_{\rho \psi{i}}^j \frac{\Delta t}{2(\eta - \theta)_i^j} \left( \frac{\delta S_v}{\delta \psi_n} \right)_i^j \right] \psi_{ni}^j \]
\begin{align*}
+ A^j_{\rho\rho i} \rho^j_{vi-1} + \left[1 - B^j_{\rho\rho i} - \frac{\Delta t}{2(\eta - \theta)} \left( \frac{\partial S_v}{\partial \rho} \right)^j_i \right] \rho^j_{vi} + C^j_{\rho\rho i} \rho^j_{vi+1} + \frac{\Delta t}{(\eta - \theta)} S^j_{vi}
\end{align*}

(23)

where $B^j_{\rho\rho i}$, $A^j_{\rho\rho i}$, $B^j_{\rho\rho i}$, and $C^j_{\rho\rho i}$ are linearized C-N coefficients related to the transport terms of Equation (7) and $\Delta t$ [s] is the time step. Here $\Delta t = 1.2$ s and was chosen after testing the model at $\Delta t = 0.3$ s and $\Delta t = 0.6$ s to ensure no degradation in model performance or solution stability at the larger time step.

3.2 Upper Boundary Conditions

The upper boundary condition on heat and vapor transfer are formulated in terms of the surface energy balance and, except for the latent heat flux, is identical to Massman’s [2012] upper boundary condition.

\[
\epsilon(\theta_0) Q^\downarrow_R(t) = \epsilon(\theta_0) \sigma T_{K0}^4 + \rho_a c_p \rho_a \rho H \left[ T_0 - T_{amb}(t) \right] + L_v E_0 + G_0
\]

(24)

where the ‘0’ subscript refers to the surface and the terms from left to right are: the incoming or down welling radiant energy, $Q^\downarrow_R(t)$ [W m\(^{-2}\)], absorbed by the surface, which is partitioned into the four terms (fluxes) on the right side of the equation, the infrared radiation lost by the surface, the surface sensible or convective heat, the surface latent heat, and the surface soil heat flux. $Q^\downarrow_R(t)$ and $T_{amb}(t)$ are functions of time and are prescribed externally as discussed in Massman [2012]. The soil surface emissivity, $\epsilon(\theta_0)$ [dimensionless], is a function of soil moisture and is taken from Massman [2012], as is surface heat transfer coefficient $C_H$ [m s\(^{-1}\)]; and $\sigma$ is the Stefan-Boltzmann constant.

The surface evaporation rate, $E_0$ [kg m\(^{-2}\) s\(^{-1}\)], is parameterized as

\[
E_0 = h_{\rho0} C_E \left[ \rho_{v0} - \rho_{vamb}(t) \right]
\]

(25)
where $h_{s0} \equiv a_{w0} = \exp([M_w \psi_n \psi_{n0}]/[RT_{K0}])$ [dimensionless] is the ‘surface humidity’, here modeled as the water activity at the surface using the Kelvin Equation; $C_E$ [m s$^{-1}$] is the surface the transfer coefficient, an adjustable model parameter but one that can reasonably be assumed to be between about $10^{-4}$ m s$^{-1}$ [Jacobs and Verhoef 1997] and $10^{-3}$ m s$^{-1}$ [Massman 2012]. Finally, in the case of the laboratory experiments of Campbell et al. [1995], $\rho_{v,amb}(t)$, like $T_{amb}(t)$ and $Q_{R}(t)$, is an external forcing function at the soil surface. The present formulation of $E_0$ results from combining and adapting the expressions for the potential evaporation rate for soils developed by Jacobs and Verhoef [1997] and Equation (9.14) of Campbell [1985]. For this formulation the surface relative humidity, $h_{s0}$, is the surface property that constrains or reduces the surface evaporation $E_0$ to less than the potential rate.

The upper boundary condition on soil water is $(\partial \theta / \partial z)_0 = 0$, which when employed with the WRC, Equation (15), yields the following upper boundary condition on the conservation of soil moisture, Equation (6):

$$
\left( \frac{\partial \psi_n}{\partial z} \right)_0 = \left( \frac{D_{\theta T}}{D_{\theta \psi}} \right)_0 \frac{G_0}{\lambda_{s0}}
$$

The boundary forcing functions $e_{v,amb}(t)$ [Pa] (the ambient vapor pressure), $T_{amb}(t)$, and $Q_{R}(t)$ are taken from Massman [2012], which in turn were adapted to the laboratory data of Campbell et al. [1995]. They take the following generic form:

$$
V(t) = V_i e^{-t/\tau} + V_f (1 - e^{-t/\tau})
$$

where $V_i$ is the value of the function at the beginning of the soil heating experiment, $V_f$ is the value of the function at the end of the experiment, and $\tau$ [s] is a time constant of the heating source, which varies with each individual soil heating experiment. $\rho_{v,amb}(t)$ is obtained from
\( e_{amb}(t) \) and \( T_{amb}(t) \) using the ideal gas law.

3.3 Lower Boundary Conditions and Initial Conditions

As with the companion model [Massman 2012], a numerical (or extrapolative or “pass-through”) lower boundary condition [Thomas 1995] is also used for the present model. Analytically this is equivalent to assuming that the second spatial derivative (\( \partial^2 / \partial z^2 \)) of all model variables is zero at the lower boundary. It is used here for the same reason as with the previous model: principally for convenience because it is likely to be nearly impossible to specify any other the lower boundary condition during a real fire. The boundary condition on the advective velocity is \( u_{vl} = 0 \) at the bottom boundary, which is also the same as with Massman [2012] and Campbell et al. [1995]. Further discussion on the model’s lower boundary conditions can be found in Massman [2012].

Except for the initial value of \( \psi_{in} \) (or \( \psi_{n, in} \)), all initial conditions (soil temperature and moisture content), which are assumed to be uniform throughout the soil column for each soil type and heating experiment, are taken directly from Campbell et al. [1995]. The initial value for \( \psi \) is obtained by inverting (solving for it using) the WRC after inputing the initial values for soil temperature and moisture content. Consequently, \( \psi_{n, in} \) can vary with the specific WRC.

4. Results

4.1 Recalibration of Observed Volumetric Soil Moisture

In the original soil heating experiments of Campbell et al. [1995] soil temperatures were measured with copper-constantan thermocouples at the sample surface and at 5, 15, 25, 35, 65, and 95 mm depth and changes in soil moisture were obtained by gamma ray attenuation at the same depths (except the surface). The moisture detecting system was linearly calibrated for each experimental run between (a) the initial soil moisture amounts, which were determined gravimetrically beforehand, and (b) the point at which the sample was
oven-dried (also determined before the heating experiment) where \( \theta = 0 \) is assumed. But
oven-drying a soil will not necessarily remove all the liquid water from a soil, i.e., a soil
can display residual water content, \( \theta_r \), after oven-drying. Consequently, the soil moisture
data obtained and reported by Campbell et al. [1995] show negative soil moistures at the
time the soil dryness passes outside the oven-dry range. Massman [2012] commented on this
issue. With the present study, all volumetric soil moisture data were first adjusted (using a
linear transformation) to rescale the observed soil moisture, \( \theta_{\text{observed}} \), so that the values of
\( \theta_{\text{observed}} < 0 \) became \( \theta_{\text{observed}} \approx 0 \). This re-scaling had very little impact on any values of
\( \theta_{\text{observed}} \) except those asymptotic data where \( \theta_{\text{observed}} < 0 \). Furthermore, this re-calibration
is reasonable so long as the original calibration was linear and based on a Beer’s Law type
extinction coefficient (which would be linearly related to the logarithm of the attenuation of
gamma ray intensity).

4.2 Model Performance

The present model is evaluated against five of Campbell et al.’s [1995] soil heating exper-
iments: (1) Quincy Sand, which has an initial soil moisture content = \( \theta_{\text{in}} = 0.14 \text{ m}^3 \text{ m}^{-3} \);
(2) Dry Quincy Sand with \( \theta_{\text{in}} = 0.03 \text{ m}^3 \text{ m}^{-3} \), (3) Dry Palousse B with \( \theta_{\text{in}} = 0.07 \text{ m}^3 \text{ m}^{-3} \),
(4) Wet Palousse B with \( \theta_{\text{in}} = 0.17 \text{ m}^3 \text{ m}^{-3} \), and (5) Wet Boulder Creek with \( \theta_{\text{in}} = 0.22 \text{ m}^3 \text{ m}^{-3} \). But here the focus is principally on Quincy Sand and Wet Palousse B. Most of the
major conclusions regarding the present model can be drawn from these two experiments and
including Quincy Sand here also benefits comparisons with Massman [2012], who also tested
his equilibrium model against the Quincy Sand data. But in addition to a general assessment
of model performance, these two experiments also serve as vehicles for a sensitivity analysis
of the key physical processes and parameterizations discussed in the previous two sections.
Thus, the Quincy Sand experiment is also used to explore the importance of the infrared
component of the soil thermal conductivity \( [\lambda_s^{\text{ir}}]: \text{ Equation (14)} \) and the Wet Palousse B
experiment is also used to evaluate the potential contribution of the dynamic residual soil moisture \([\theta_r(t, n, T_K)]\): Equations (15), (16) and (17)\] to the model’s performance. Finally, because the HCF [Equation (17)], is central to the overall performance of the model it is discussed in more detail a separate section following the Quincy Sand results.

4.2.1 Quincy Sand

Figure 1 compares the measured (symbols) and modeled (lines) of soil temperature during the Quincy Sand heating experiment for two model runs with different values of \(R_p\). Increasing \(R_p\) to 4 mm (dashed lines) over the default value of 1 mm (solid lines) increases the infrared component of the soil thermal conductivity, \(\lambda_s^{[2]}\). Because the model’s performance was not enhanced by the inclusion of either the residual soil moisture, \(\theta_r\), or film flow, \(K_{R}^{\text{film}}\), neither are included as part of the Quincy sand simulations. The colors indicate the depths (mm) of the experimental and model data. (Note: the same color is also used to denote to same depth for both the model and observed data in Figures 2, 3, and 4 below.)

The corresponding measured and modeled soil moisture is shown in Figure 2. These two figures indicate that the present model produces results that are similar to both the original Campbell et al. (1995) model and the observations. But they also indicate that the \(R_p = 4\) mm simulation is superior to \(R_p = 1\) mm. Furthermore, comparing these two Figures with their counterparts in Massman [2012] clearly indicates that the non-equilibrium model is a substantial improvement over the (older) equilibrium model, a conclusion that is easily confirmed by comparing Figures 4 through 7 below with their equivalents in Massman [2012].

Figure 3 is a plot of the data trajectory (observed soil temperatures vs observed soil moistures for all the monitored depths). The model’s solution trajectories (for the same depths and the two \(R_p\) values) are shown in Figure 4. Comparing these two figures suggest that the model does a reasonable job of capturing the rapid vaporization of soil water at temperatures between 70 C and 90C (at least at the depths below about 10 mm). But the
present configuration of the model does not do as well at capturing the amplitude (amount) of the recondensing moisture ahead of the rapid drying nor the duration (or width of the amplitude) of the recondensation. Furthermore, when compared with observations (Figure 3) the model does not fully capture the amount of unevaporated soil moisture that remains at temperatures $\geq 150$ C. In this regard neither version of the model is significantly different from the other. But some of this “lack of precision” with the soil moisture simulation results in part by how the HCF was calibrated and will be examined in more detail in the next section.

Figure 5 compares the vertical profiles of the soil temperatures at the end of the laboratory experiment with those at the end of the two numerical simulations and Figure 6 makes a similar comparison for the volumetric soil moisture content. These figures also include the modeling results synchronized in time and space with the observations, which are included to make the model output more directly comparable to the observations. [Note: the final vertical profiles obtained from the laboratory experiment are not coincident in time with the measurements made at any other depth. This is a consequence of the experimental design, which required several minutes to complete one vertical scan for soil moisture.] Figure 5 clearly indicates that the $R_p = 4$ mm simulation does a better job of capturing the final temperature profile than does the $R_p = 1$ mm simulation.

The difference between the these two model simulations is less obvious with Figure 6. The final soil moisture profiles shown in this figure can be used to estimate the percentage of soil moisture evaporated and lost from the soil column at the end of the 90-minute experiment, $E_{\text{loss}}$. The laboratory observations suggests 31% was lost. The ($R_p = 1$ mm; red) model simulation indicates a 31.4% loss and the corresponding (red) synchronized-model yielded a 33.8% loss. The ($R_p = 4$ mm; blue) model simulation indicates a 34.6% loss and the corresponding (blue) synchronized-model yielded a 34.2% loss. Because the fully sam-
pled and synchronized model results give somewhat different percentage loss it is possible to conclude that the laboratory estimate of evaporative loss is likely imprecise because it is poorly resolved in time and space. So exact agreement between model and observations is, in general, unlikely. But it is possible to use the model itself to estimate the uncertainty in the observed moisture loss that is caused by this under-sampling. This is done by comparing and synthesizing all fully sampled model estimates of $E_{loss}$ with all the corresponding synchronized model estimates. For the five experiments studied here this uncertainty, expressed as a fraction, is about ±0.03.

For the present Quincy Sand experiment this yields a fractional $E_{loss} = 0.31 ± 0.03$, which in turn suggests that both the present simulations ($R_p = 1$ mm and 4 mm) provide quite good estimates of the total water loss as well as the final profiles of soil moisture and may in essence really be indistinguishable. But combining these soil moisture results with the final temperature profiles suggest that the $R_p = 4$ mm simulation is the preferred. In addition, the present model results are significantly better than the equilibrium model, which found that no water was lost during the experiment, a clearly implausible result! [Rather than actually transporting the evaporated water out of the soil column, the equilibrium model “pushed” the moisture deeper into the soil ahead of the evaporative front as discussed in the Introduction.] On the other hand, despite the fact that the present estimates of evaporative loss are clearly a major improvement over the equilibrium results, both the equilibrium and non-equilibrium model solutions produce a sharply delineated advancing drying front, which is reminiscent of a Stefan-like or moving-boundary condition problem (e.g., see Whitaker and Chou 1983-1984 or Liu et al. 2005). So neither simulation actually captures the final observed moisture profile behind the drying front.

Figure 7 shows the final (90-minute) modeled profiles of (a) the ($R_p = 1$ mm) soil vapor density [$\rho_v(z)$], equilibrium vapor density [$\rho_{ve}(z)$], and the condensation term [$K_c(z)\rho_v(z)$],
used with the non-equilibrium model source term, $S_v$: Equations (10) and (11)] and (b) the ($R_p = 4$ mm) soil vapor density $[\rho_v(z)]$. The solid lines are model simulations with $R_p = 1$ mm; the dashed red line corresponds to the $R_p = 4$ mm simulation. (For the sake of simplicity only one curve is shown for $R_p = 4$ mm simulation.) The maximum soil vapor density occurs at about $40$ mm where the evaporative source term is greatest, i.e., where $\rho_{ve}(z) - K_c(z)\rho_v(z)$ is maximal, and where the moisture gradient is steepest, which is just ahead of the drying front (Figure 6). Furthermore, the $\rho_v$ profile suggests that there are both upward and downward diffusional fluxes of vapor away from the maximal evaporative source. The upward-directed flux escapes through the soil surface and into the ambient environment of the laboratory (the surface evaporative flux) and the downward-directed flux eventually recondenses below of the dry front. The equilibrium model, on the other hand, produced virtually no vapor gradient within the dry zone thereby contributing to the model’s inability to allow any moisture to escape (evaporate) from the modeling domain. Unfortunately, there are no observations with which to check either models’ predictions of vapor density. But, although both models’ results tend to agree that within the dry zone where temperatures exceed the critical temperature for water ($= 373.95$ C) there should be a single phase fluid that is significantly denser than water vapor near STP (e.g., Pakala and Plumb 2012), the non-equilibrium model does predict a more realistic vapor gradient than the non-equilibrium model.

If there is an implausibility with the present model it might be the soil vapor pressure, $e_v$, as shown in Figure 8. With either model simulation $e_v$ at the top of the soil column is between about 3 standard atmospheres ($\approx 300$ kPa). This is a bit unexpected because pressure at the open end of the column might be expected (at least by this author) to be close to equilibrium with the ambient pressure ($\approx 92$ kPa). Although there are no data against which to check this result, there are other modeling results that lend some support to the
present predictions for $e_v$. First, (Figure 5 of) Udell’s [1983] steady state model of a sand-water-steam system heated from above indicates that the environment within the modeling domain is likely to be super-saturated and that at a minimum $e_v$ is greater than $P_{atmos}$ by $\approx 5\%$, but (depending on the algorithmic treatment of the saturation vapor pressure and the exact value of $P_{atmos}$ he used for his simulations) it is also plausible to expect that $e_v \approx (2 - 5)P_{ST}$. (Note that for Udell’s [1983] simulations the maximum model temperature was about 180 C and that he also modeled advective velocity using Darcy’s law.) Second, two different models of heated cement [Dayan 1982 and Dal Pont et al. 2011] indicate that near the top surface of the model domain $e_v$ can display values of $\approx (2 - 15)P_{ST}$. The overall similarities between these three earlier models and the present non-equilibrium model make it impossible to completely invalidate the present model’s predictions for $e_v$. Furthermore, the non-equilibrium model imposes no particular constraint on $e_v$ — it is calculated using the ideal gas law and the profiles of vapor density and temperature, both of which appear plausible. Consequently, the somewhat surprising result shown in Figure 8 appears to be a natural consequence of the physics underlying the basic model equations: the conservation of mass and thermal energy.

**4.2.2 HCF – Quincy Sand**

Figure 9 shows the hydraulic functions $K_{cap}^H$, $K_{film}^H$, $|K_n^s|$, $|K_n|$, and $|K_{surf}^n|$ as functions of soil moisture for the model simulation with $R_p = 1$ mm. (Recall that the components of the hydraulic diffusivity are all negative, so this figure reflects only their absolute value, not their sign). $K_{film}^H$ [assuming $F_w = 10^3$, Equation (19)] was calculated after the model run using the model’s solution for $T_K$, $\theta$, and $\psi$. Because $K_{film}^H$ did not actually contribute anything to any of the model runs for any of the five soil heating experiments (even for $F_w = 10^4$), the present approach for evaluating it is sufficient. Consequently although $|K_n|$ shown here is more properly termed $|K_{cap}^n|$, this distinction is rendered moot for the present study.
The values for $m$ and $n$ of $K_{\text{cap}}^R$, Equation (17), were $m = 0.26$ and $n = 1.80$ and were obtained by subjectively optimizing the model (with $R_p = 1$ mm) to fit the data shown in Figures 1, 2, 5, and 6. In other words, it is possible to improve the model’s performance for either the temperature data or the soil moisture data individually, but not simultaneously. Any improvement in one set of observations ($T_K$ or $\theta$) comes at the expense of the model’s performance for the other variable. As a consequence of this optimization the model’s ability to fully capture the amplitude (amount) of soil moisture that evaporates and recondenses ahead of the drying front [Figures 3 and 4] has been sacrificed to improve the simulation of the soil temperatures. So the present numerical solution is a compromise between trying to fit two sets of data with a single “best” parameterization of Assouline’s [2009] $K_{\text{cap}}^R$. Similar compromises were required for other heating experiments as well. These particular optimal values for $m$ and $n$ were also found valid for the Bouldercreek soil experiments; but for the Paloussé B soil the optimal values were $m = 0.29$ and $n = 1.82$.

Unfortunately, there are no independent confirmations for any values of $m$ and $n$ because no soil hydraulic conductivity data were (or ever have been) obtained for any of the soil samples used by Campbell et al. [1995]. Nor are any data likely at this point in time because the original samples were destroyed years ago (G. S. Campbell, personal communication, 2015). At the time of these laboratory experiments the basic assumption that the original researchers made was that the heating and evaporation rates would be so fast as to preclude any (presumably much slower) liquid water transport associated with gradients in soil water potential. The present results appear to invalidate that assumption.

Although it is undeniably true that the present model is an improvement over the equilibrium model, the inclusion of the HCF within this non-equilibrium model (and its lack of inclusion in the equilibrium model) makes it difficult to conclude unambiguously that the improvement over equilibrium model is the sole consequence of the non-equilibrium assump-
tion. But the non-equilibrium model was tested in a mode that basically “turned off” $K_R$ and reduced $K_{n, surf}^{surf}$ by several orders of magnitude and still it yielded a solution (not shown) that simulated the soil temperatures and soil moisture observations better than did the equilibrium model. Furthermore, the non-equilibrium assumption will always remain superior to the equilibrium assumption for dry or extremely dry soils. Nonetheless, it remains unknown how much, if any, improvement in the equilibrium model’s performance is possible with the inclusion of a HCF.

Finally it should also be pointed out that, unlike $K_R^{film}$, $K_{n, surf}^{surf}$ does contribute positively and significantly to the present model’s performance because without it the model can become unstable for very dry soils. Therefore, $K_{n, surf}^{surf}$ is a significant factor in the HCF’s overall contribution to the performance of the non-equilibrium model.

4.2.3 Wet Palousse B

Figure 10 compares the measured (symbols) and modeled (lines) of soil temperatures during the Wet Palousse B heating experiment with the model run that includes the residual soil moisture, $\theta_r$ (dashed), and that which does not (solid). Figure 11 compares the measured (symbols) and modeled (lines) of soil moisture during the same experiment. Figure 12 shows the two models’ solution trajectories, where the dashed line is the model run that includes $\theta_r$ and the solid line does not. These results suggest that the inclusion of the $\theta_r$ in the model does not influence temperatures very much, but that the soil moisture dynamics are much better portrayed by the model with $\theta_r$ than without it. In fact, the main difference between these two simulations is that the soil moisture that evaporates and recondenses ahead of the drying front is much less for the model that includes $\theta_r$ than that which does not (Figure 12). The significance of this aspect of the model’s performance is demonstrated in Figure 13, which, by comparing the observed Palousse B Wet trajectory data with the data from the $\theta_r$-configured model, demonstrates that the $\theta_r$-configured model reproduces the observations.
far more closely than does the model configured without $\theta_r$. The next two figures, Figures 14 and 15, compare the final modeled and observed profiles of temperature and moisture for the Wet Palouse B experiment. These last two figures demonstrate (even more clearly than the previous figures) that including the dynamic soil moisture, $\theta_r$, has very little effect on the modeled soil temperatures, but that it does has a significant and positive effect on the modeled soil moistures. The predicted soil water loss from the $\theta_r$-configured model is almost exactly in agreement with the observed value of 28.8% and the modeled moisture profile below the drying front is also in almost perfect agreement with the observed data. The model without $\theta_r$ allows much more evaporated moisture to diffuse downward and to recondense ahead of the drying front than does the $\theta_r$ model; thereby underestimating the water loss by about half and significantly overestimating the amount of soil water in lower portion of the profile.

### 4.3 Further Sensitivity Analyses

The Quincy Sand and Palouse B results in general confirm that the non-equilibrium model’s performance is enhanced (and quite significantly) with the incorporation of liquid water transport (HCF) and that its performance is sensitive to (and can be improved by) either or both (a) the infrared thermal conductivity, $\lambda_s$ [2], through the volumetric pore radius, $R_p$, and (b) the dynamical residual soil moisture, $\theta_r(\psi, T_K)$. Although these last two aspects of the present model may not apply equally to any given soil, there seems little doubt that they should be considered as potentially quite important for modeling soil moisture and temperature dynamics during heating of soil during fires.

The remainder of this section is a summary of various (secondary, but important) model sensitivity analyses performed with all soil heating experiments. The ultimate intent here is to shed light on which physical process are relatively more important and to provide some guidance for further research.
4.3.1 The Source Term, Thermal Conductivity, and Surface Evaporation

There is little doubt that the present non-equilibrium model is an improvement over the equilibrium model of Massman (2012) and central to the success of the present model is the functional parameterization of the source term, $S_v$, and the related condensation coefficient, $K_c(T_K, \psi_n)$. Basically $K_c$ was required to maintain model stability especially at high temperatures; without it the model was unstable and the dynamic between moisture and vapor was non-physical. Regarding $K_c$, the model is weakly sensitive to the choice of the surface evaporation/condensation activation energy, $E_{av}$, providing it does not vary much outside the range of $30 \text{ kJ mol}^{-1} \leq E_{av} \leq 40 \text{ kJ mol}^{-1}$. On the other hand, from a systems perspective it is very difficult to infer much about the details of the (high-temperature) physical processes associated with $K_c$ or of the generality/universality of $E_{av}$, other than their apparent existence and utility to the present model. The most effective value for the scaling parameter, $S^{[M]}_v$, was within the range of about 0.5 to 1. The Novak model of the source term, $S^{[N]}_v$, also required the same $K_c$, but the additional temperature dependency of $S^{[N]}_v$ over $S^{[M]}_v$ forced the soil moisture to evaporate at slightly lower temperatures (therefore sooner) than shown in Figure 4 for the Quincy Sand $S^{[M]}_v$-configured model. $S^{[N]}_v$ also eliminated an initial transient/instability (not shown) that occurred with the Quincy Sand $S^{[M]}_v$ solution. Otherwise, the differences between $S^{[N]}_v$ and $S^{[M]}_v$ were not significant.

It should not be surprising that the model is sensitive to soil thermal conductivity, $\lambda_s$; but it was somewhat surprising that the model is as sensitive to $\lambda_s^{[2]} [R_p]$ as it is. Both Durany et al. [2010] and Massman [2012] found that $\lambda_s^{[2]}$ only contributed for relatively porous soils, i.e., $R_p > 10^{-3}$ m. In the present study the model temperatures could often be “fine tuned” (improved) even for relatively small values of $R_p$. Therefore, in general, it seems unwise to ignore $\lambda_s^{[2]}$, at least when modeling soil heating during fires.

The most important parameter controlling surface evaporation rate is the surface transfer...
coefficient $C_E$, to which the model is reasonably sensitive. In particular (and similar to Massman’s 2012 results), the best (maximal) values of $C_E$ were universally about $10^{-3}$ m s$^{-1}$ and values much above this caused the model to become unstable. Values well below these values (and closer to the theoretical value of $10^{-4}$ m s$^{-1}$) did not produce results much different than those resulting from $C_E = 10^{-3}$ m s$^{-1}$. Nevertheless, $C_E$ does play a weak role in determining the soil surface temperature, $T_{K0}$, and therefore can influence the magnitude of the surface convective heat flux – Equation (24) – and $\lambda^{[2]}_{s0}(T_{K0})$, which can influence the soil temperatures to some depth below the soil surface.

4.3.2 Water Retention Curves and Hydraulic Conductivity Functions

The two other WRCs tested for model performance were Groenevelt and Grant [2004] (GG04) and Fredlund and Xing [1994] (FX94). But prior to implementing them in the model they were both calibrated to be numerically similar near the dry end ($\theta \leq 0.03$) of Equation (15). Their performance was initially tested with the Assouline [2009] HCF, Equation (17). In addition, and as listed in section 2.2.5, besides Equation (17) four other HCFs were also tested. Although not all pairings of WRCs and HCFs were tested against every heating experiment, the following conclusions seem relatively robust: (1) once calibrated to the present data set, the Brooks and Corey [1964] HCF performed at least as well as Equation (17), but with only one parameter rather than two, while the other three HCFs gave somewhat less satisfying results and sometimes would even produce an instability; (2) the FX94 WRC often produced an instability, but its performance was also somewhat dependent upon which HCF was used with it; and (3) the GG04 WRC and associated HCFs did perform better the FX94, but overall, it did not perform as well as the present model configuration with Equation (15) for the WRC and Equation (17) for the HCF. In general the only guidance offered here is that some care must be given to choice of WRCs and HCFs because the modeling results can be quite dependent upon the choices made.
Universal to all of HCFs tested here is $K_{n}^{surf}$ [i.e., $V_{θ,surf}$ and its scaling parameter $D_{θs0}$: see Equations (20) and (21)], which as explained above was incorporated into the hydraulic diffusivity, $K_{n}^{*}$ [see Equation (6) and the related discussion]. The model’s performance was relatively insensitive to the exact value of $D_{θs0}$ and $D_{θs0} \approx 10^{-10}$ m$^2$s$^{-1}$ is as good a default value as any other value. Furthermore, the inclusion of $V_{θ,surf}$ (soil moisture movement associated with water molecules “hopping or skipping” along the soil and water surfaces) did provide model stability when the soil moisture reached extremely low values. Including film flow, $K_{r}^{film}$ – Equations (18) and (19), brought no discernible benefit to the model’s performance. But that does not mean that under more benign conditions that it would not. Consequently it seems reasonable to further investigate the possibility of film flow when modeling less extreme conditions.

4.3.3 Different Soils with Different Initial Conditions

Of the remaining three heating experiments only Wet Boulder Creek, which had an initial soil saturation level of about 50%, showed anything unexpected. In general, the model was able to capture the observed soil temperatures and temperature dynamics extremely well, even better than shown in either Figure 5 (Quincy Sand) or Figure 14 (Palouse B). But, regardless of any adjustments to any of the model parameters the model consistently underestimated the total amount of water evaporated (by about half), thereby also overestimating the amount of recondensing water ahead of the drying front (see the Wet Palouse B model run without $θ_r$, Figure 15, as an example). On the other hand, the model was able to capture the complete drying ($θ = 0$) of the top 30 mm of the soil column during the Wet Boulder Creek experiment, whereas the Quincy Sand (Figure 6) and Palouse B (Figure 15) experiments indicated some residual soil moisture ($0 < θ < 0.025$) within the model’s predicted dry zone. The model’s performance was noticeably degraded when $θ_r$ was included and it was also quite sensitive to the choice of $R_p$, so much so that best model fit to temperatures required
that $R_p = 0$, i.e., that $\lambda_s^{[2]}$ be excluded from the model. Nonetheless, the cause for this unexpected divergence between the modeling and observed soil moisture during this heating experiment is not understood. For the present it can only be surmised that the model’s description of evaporative dynamics of soil moisture, and possibly the transport of water (both liquid and vapor), is still incomplete.

4.3.4 The advective velocity, $u_{vl}$

Unlike with the companion model [Massman 2012], the present model did not require reducing the magnitude of $u_{vl}$ in order to maintain model stability, which again reinforces the impression that the present non-equilibrium model is an improvement over equilibrium model. Nonetheless, the present model can produce extraordinary gradients of vapor density and vapor pressure, which begs the question of whether such gradients could induce other types of mass transport than that captured by the present formulation of $u_{vl}$, Equation (9). This was tested by using a model for Darcy’s Law type formulation based on the assumption that the advective velocity is proportional to the vapor pressure gradient ($u_{vl} \propto -\partial e_v/\partial z$). This formulation was tested by incorporating it into the model (excluding $\theta_r$). But the model became unstable because mathematically it was strongly hyperbolic, rather than predominantly parabolic. Further modeling development and parameterization of $u_{vl}$ and vapor transport in general are well beyond the intent of the present study. But it is still possible to conclude that such exploration is warranted and could help improve model performance. Finally, it should be noted that the model’s performance was degraded when $u_{vl}$ was excluded from the model.

5. Summary and Recommendations

This study has developed and tested a non-equilibrium (liquid to vapor phase change) model for simulating heat and moisture flow in soils during fires; but the model does assume thermal equilibrium. By and large the simulations of soil temperature and moisture are
not only credible, but often quite good. In general, all model results showed a significant improvement over all comparable results from the companion equilibrium model of Massman [2012].

The principal reason for the present model’s success is the incorporation of a dynamic condensation coefficient, $K_c$ (parameterized as a function of temperature and soil water potential), into the non-equilibrium evaporative source term, $S_v$; both of which are modeled after the Hertz-Knudsen Equation. Physically $K_c$ suppressed condensation in favor of evaporation at high temperatures and soil water potentials, which in turn insured model stability. Furthermore, the non-equilibrium assumption also seemed to have improved the parameterization (and performance) of the mass transport associated with the advective velocity, $u_{vel}$, relative to the model’s of Massman [2012] and Campbell et al. [1995]. The model’s performance was further and significantly enhanced by the inclusion of a hydraulic conductivity function (HCF) for liquid water transport, which was calibrated here by “fitting” the HCF parameters to ensure that the model optimally reproduced the observed temperature and moisture dynamics. This fitting procedure was necessary because no data are (nor will be) available for the soil samples used in the laboratory heating experiments [Campbell et al. 1995]. Another important (and novel) feature of the model is the inclusion of a dynamic residual soil moisture $\theta_r$, also parameterized as a function of temperature and soil water potential, which is introduced into the model in an attempt to capture the long evaporative tail that seems to require temperatures well beyond 100 C in order to evaporate at all. Physically $\theta_r$ is intended to represent the strongly bound soil moisture, which for the present purposes is conceptualized as a mono-layer. Including $\theta_r$ was sometimes, but not always, beneficial to model performance. So it seems worthy of further consideration and possible refinement in any future studies of a similar nature. Finally, the model is also sensitive to the thermal infrared radiation component to the soil’s thermal conductivity [$\lambda_s^{[2]}$: Equation (14)], which
increases the thermal conductivity within the pore space of the soil as temperature increases. It is recommended that this term also be included and further tested when evaluating any future models describing the heating soils to high temperatures.

In general, the model simulates the observed soil temperatures quite well. It is often slightly less precise for soil moisture and the best simulations were usually a compromise between faithfully representing the observed soil temperatures or the observed soil moistures. Nonetheless, the model does capture reasonably well many observed features of the soil moisture dynamics, viz., it simulates an increase in soil moisture ahead of the drying front (due to the condensation of evaporated soil water at the front) and the hiatus in the soil temperature rise during the strongly evaporative stage of the soil drying. Furthermore, the model also captures the observed rapid evaporation of soil moisture that occurs at relatively low temperatures (50-90°C), as well as some aspects of the long evaporative tail associated with strongly bound soil moisture. But, the model also displays a tendency to predict a greater depth of the drying front than suggested by the observations.

Sensitivity analyses (SAs) were also performed with different formulations for the water retention curve, soil hydraulic conductivity function, one variant of the present evaporative source term, $S_v$, and different soil types with different initial conditions. The principal conclusion from these SAs is that some care (and testing) must be given to the selection of the WRC and HCF, as not all of them performed equally well. Some further investigations into the modeling benefit of film flow as part of the HCF also seems warranted. The two forms of $S_v$ tested here performed about the same. And the model’s performance (at least for soil moisture) was poorest compared to the experiment with the highest initial soil moisture content. No obvious explanation for this ‘under-performance’ could be found, so it seems worthwhile to further test the model for high initial saturation conditions. Finally, it is important to test the present model’s performance and its associated parameterizations.
(particularly the WRC and HCF) against laboratory data and field data associated with
daily cycles of soil heating and moisture transport.

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Quincy Sand --- Temperature Dynamics

symbols = data
solid lines: $\lambda_s^{[2]}$ with $R_p = 1$ mm
dotted lines: $\lambda_s^{[2]}$ with $R_p = 4$ mm

Figure 1:
Figure 2:
Figure 3:
Figure 4:
Figure 5:
Final Profile Soil Moisture

- Observed Data
- Model: $\lambda_s^{[2]}$ with $R_p = 1$ mm
- Synchronized Model
- Model: $\lambda_s^{[2]}$ with $R_p = 4$ mm

Figure 6:
Figure 7:
Figure 8:
Figure 9:
Figure 10:
Figure 11:
Figure 12:
Figure 13:
Figure 14:
Final Profile Soil Moisture

Observed Data
Model: $\theta_r$ not included
Synchronized Model
Model: $\theta_r$ included
Synchronized Model

Figure 15:
Figure Captions

Figure 1. Comparison of measured (symbols) and modeled (lines) soil temperatures during the Quincy Sand heating experiment. Neither simulation includes a dynamic residual soil moisture term, $\theta_r$. The solid lines are for a model simulation with $R_p = 1$ mm; the dotted lines correspond to a simulation for $R_p = 4$ mm. Note as $R_p$ increases the infrared portion of the soil thermal conductivity, $\lambda_s^{[2]}$, also increases, in accordance with Equation (14). To compare with the equilibrium model see Figure 2 of Massman 2012.

Figure 2. Comparison of measured (symbols) and modeled (lines) soil moisture contents during the Quincy Sand heating experiment. Neither simulation includes a dynamic residual soil moisture term, $\theta_r$. The solid lines are for a model simulation with $R_p = 1$ mm; the dotted lines correspond to a simulation with $R_p = 4$ mm. To compare with the equilibrium model see Figure 3 of Massman 2012.

Figure 3. Measured soil moisture vs measured soil temperatures for the Quincy Sand heating experiment (see previous two figures).

Figure 4. Modeled soil moisture contents vs modeled soil temperatures for the Quincy Sand heating experiment (see Figs. 1 and 2 above). The solid lines are for a model simulation with $R_p = 1$ mm; the dotted lines correspond to a simulation with $R_p = 4$ mm. This is the solution space representation of the model’s solutions, which are to be compared with the observations shown in the preceding figure, Fig. 3, as well as with the equilibrium model results shown in Figure 5 of Massman 2012.

Figure 5. Comparison of the final modeled and measured temperature profiles at the completion of the 90-minute Quincy Sand heating experiment. Because the data shown in the measured profile (black) are not precisely coincident in time, the full model results (solid red and blue lines) were sub-sampled in synchrony in time (and coincide in space) with the
observations. These time-synchronized model profiles are shown as dashed red and blue lines. To compare with the equilibrium model see Figure 6 of Massman 2012.

**Figure 6.** Comparison of the final modeled and measured moisture profiles at the completion of the Quincy Sand heating experiment. Because the data shown in the measured profile (black) are not precisely coincident in time, the full model results (solid red and blue lines) were sub-sampled in synchrony in time (and coincide in space) with the observations. These synchronized model profiles are shown as dashed red and blue lines. The observed data (black) suggest that the total water lost during the 90-minute experiment was 31% of the initial amount. The (red) model simulation indicated a 31.4% loss and the corresponding (red) synchronized-model yielded a 33.8% loss. The (blue) model simulation indicated a 34.6% loss and the corresponding (blue) synchronized-model yielded a 34.2% loss. Note there is very little recondensing soil moisture ahead of the drying front (at about 40-50 mm depth), in agreement with Figures 2 and 4 above and in contrast with the equilibrium model, Figure 7 of Massman 2012, where there was significant recondensation.

**Figure 7.** Final modeled profiles of vapor density [$\rho_v$], equilibrium vapor density [$\rho_{ve}$], and the condensation coefficient ($K_c$) modified vapor density term [$K_c\rho_v$] used with the non-equilibrium model source term, $S_v$, at the completion of the 90-minute model simulation. The three solid lines are for a model simulation with $R_p = 1$ mm; the single dotted line corresponds to a simulation with $R_p = 4$ mm. The maximum vapor density for these two simulations is between about 1.3 and 1.5 times the density of the standard atmosphere (= 1.292 kg m$^{-3}$) and is located the near position of the maximum in the vapor source term, $S_v$. This figure can be compared with the equilibrium model result: Figure 8 of Massman 2012.

**Figure 8.** Final modeled profile of vapor pressure at the end of the 90-minute model
simulation. The solid line is the model simulation with \( R_p = 1 \text{ mm} \) and the dotted line corresponds to the simulation with \( R_p = 4 \text{ mm} \). In both cases the maximum vapor pressure occurs at the soil surface and/or near the level of the maximum \( S_v \). For these two scenarios the maximum vapor pressure is about 3.2 times the pressure of one standard atmosphere (\( P_{ST} = 101.325 \text{ kPa} \)).

**Figure 9.** Example of the hydraulic conductivity, \( K_H \), and magnitude of the hydraulic diffusivity, \( |K_n| \), as functions of soil moisture, \( \theta \), for the Quincy Sand \( R_p = 1 \text{ mm} \) scenario. \( K_H \) corresponds to Assouline’s [2001] HCF for capillary flow, Equation (17), and Zhang’s [2011] model for film flow, Equation (18). Numerically \( |K_n| \) is just a rescaling of \( K_H \) (see Section 2.2.5 for further details) and \( |K_{n, surf}| \) is derived from the Gawin’s [1999] model for \( V_{\theta, surf} \) (again see section 2.2.5).

**Figure 10.** Comparison of measured (symbols) and modeled (lines) soil temperature during the Palousse B Wet heating experiment. The solid lines are for a model simulation that does not include the dynamic residual soil moisture, \( \theta_r \); the dotted lines correspond to the simulation that includes \( \theta_r \).

**Figure 11.** Comparison of measured (symbols) and modeled (lines) soil moisture content during the Palousse B Wet heating experiment. The solid lines are for a model simulation that does not include the dynamic residual soil moisture, \( \theta_r \); the dotted lines correspond to a simulation that includes \( \theta_r \). For this experiment the initial soil moisture, \( \theta_{in} \), is 0.17 m³ m⁻³.

**Figure 12.** Modeled soil moisture vs modeled soil temperatures for the Palousse B Wet heating experiment (see Figs. 10 and 11 above). The solid lines are for a model simulation that does not include the dynamic residual soil moisture, \( \theta_r \); the dotted lines correspond to a simulation that includes \( \theta_r \). This is the solution space representation of the model’s
Figure 13. Observed and modeled soil moisture vs soil temperatures (trajectories) for the Wet Palouse B heating experiment. Solid lines are observed data and the dash-dot lines are from the model that includes the dynamic residual soil moisture, $\theta_r$.

Figure 14. Comparison of the final modeled and measured temperature profiles at the completion of the 70-minute Palouse B Wet heating experiment. Because the data shown in the measured profile (black) are not precisely coincident in time, the full model results (solid red and blue lines) were sub-sampled in synchrony in time (and coincide in space) with the observations. These time-synchronized model profiles are shown as dashed red and blue lines.

Figure 15. Comparison of the final modeled and measured moisture profiles at the completion of the 70-minute Palouse B Wet heating experiment. Because the data shown in the measured profile (black) are not precisely coincident in time, the full model results (solid red and blue lines) were sub-sampled in synchrony in time (and coincide in space) with the observations. These time-synchronized model profiles are shown as dashed red and blue lines. The observed data (black) suggest that the total water lost during the 70-minute experiment was 28.8% of the initial amount. The (red) model simulation indicated a 14.7% loss and the corresponding (red) synchronized-model yielded a 15.8% loss. The (blue) model simulation indicated a 27.8% loss and the corresponding (blue) synchronized-model yielded a 29.4% loss. Note there is very little recondensing soil moisture ahead of the drying front (at about 35 mm depth), in agreement with Figures 11 and 12.