Semi-analytical 2DV perturbation model

package for iFlow

Ronald Brouwer

When using iFlow, please cite Dijkstra, Y., Brouwer, R., Schuttelaars, H., and Schramkowski, G. (Manuscript submitted to Geoscientific Model Development). The iflow modelling framework v2.4. a modular idealised processes model for flow and transport in estuaries. Additionally you may refer to this manual as Brouwer, R. (2017). *iFlow modelling framework. Semi-analytical 2DV perturbation model package for iFlow*.
Note the license obligations that come with iFlow.

Contents

1 Modules Reference 5

1.1 Hydrodynamics 6

1.2 Sediment 8

2 Introduction: domain and approach 11

I Leading-order and first-order hydrodynamics

3 Equations and ordering 15

3.1 Equations and assumptions 15

3.2 Scaling 17

3.3 Ordering & overview of the equations 22

4 Analytical solutions to the ordered equations 25

4.1 Leading order solution 25

4.2 First order solutions 30

5 Numerical implementation 45

5.1 General structure of the hydrodynamics modules 45

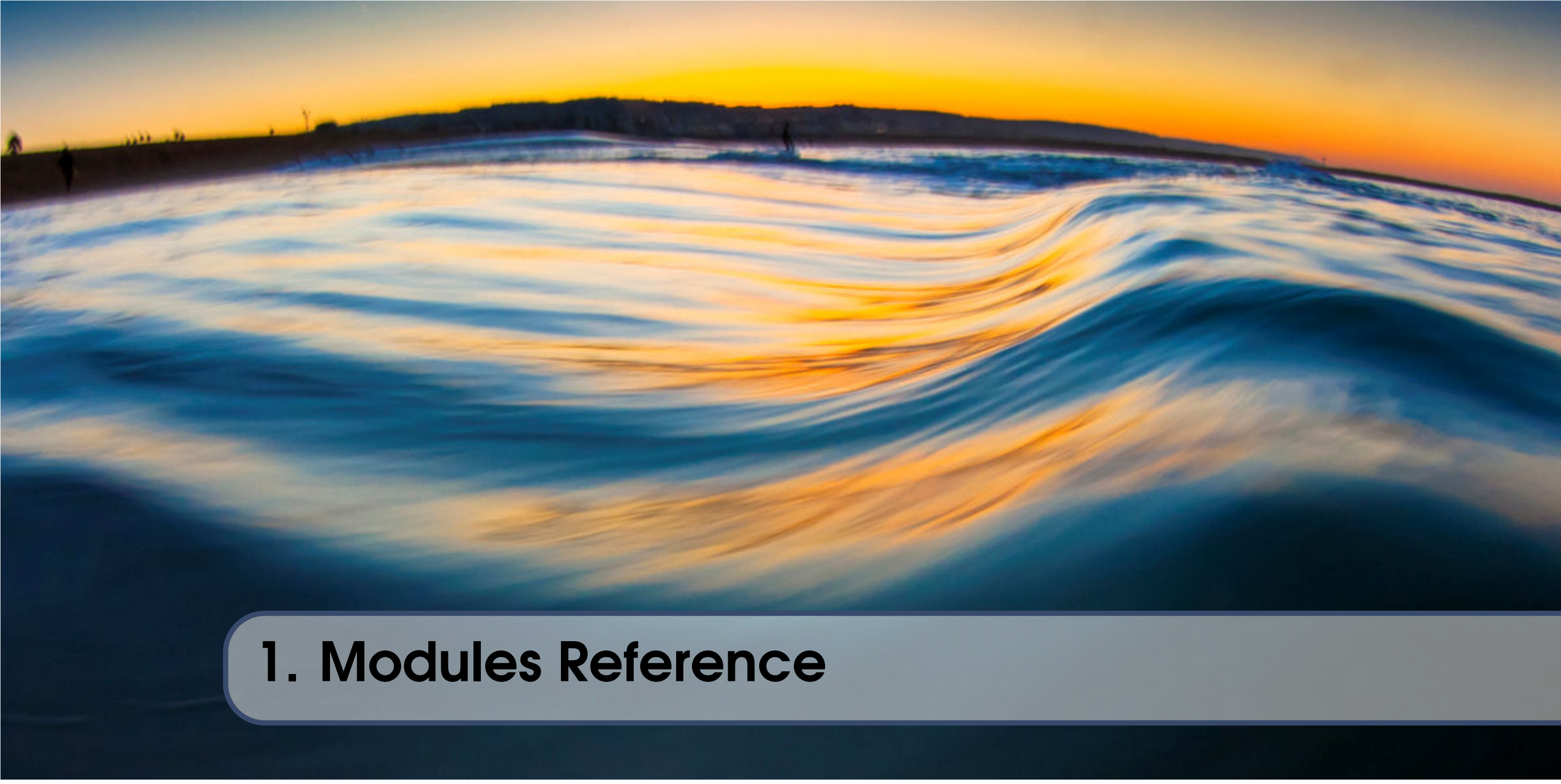
5.2 Solving the boundary value problem for the water level 46

II	Sediment dynamics

6 **Equations and ordering** 51
6.1 Equations and assumptions 51
6.2 Scaling 53
6.3 Ordering & overview of the equations 56

7 **Analytical solutions to the ordered equations** 61
7.1 Leading order solutions of sediment concentration 61
7.2 First order solutions of sediment concentration 63
7.3 Second order solution of sediment concentration due to erosion by river flow 67
7.4 Morphodynamic equilibrium condition; transport components 67

8 **Numerical implementation** 71
8.1 General structure of the sediment dynamics module 71



1. Modules Reference

This chapter provides a short overview of all modules in the package `semi_analytical2DV` and the required input and expected output. The modules have been ordered into sections for the purpose of providing structure to this chapter.

Explanation of terms and colours

Behind the input variables we will mention several data types. While some data types may be obvious, some others are explained in the table below:

Space-separated numbers	real numbers separated by one or more spaces. Do not use comma's or other markers to separate the numbers.
Grid-conform array n-dimensional	a numpy array with n (i.e. some number) or fewer (!) dimensions. More dimensions than n is not allowed. All axes should be grid conform. That means that the length of a dimension should either be 1 or equal to the size of the corresponding grid axis. If n is larger than the grid size, the length of this axis is free. Note that a single number counts as a grid-conform array.
General n-dimensional	either a grid-conform array or a numerical or analytical function. In both cases they may n (i.e. some number) or fewer dimensions.
iFlow grid	a grid variable with underlying subvariables as described in the manual (Dijkstra, 2017b)

The cells with input variables have been colour-coded to indicate whether the variable is likely to be given in the input file, computed by another module or given in the configuration file. By the very nature of iFlow this is only indicative and depends on the modules used. As an example, almost any variable given in the input file may be used as a variable in a sensitivity analysis. It then becomes an input parameter of the sensitivity analysis module in the input file. The sensitivity analysis module delivers it to the module that uses this variable.

(green)	Likely a parameter in the input file
(orange)	Either in the input file or from another module
(red)	Likely a parameter computed by another module
(grey)	Likely a constant in the configuration file `src.config`

1.1 Hydrodynamics

1.1.1 HydroLead

Leading-order hydrodynamics using a semi-analytical perturbation model. See Part I of this manual.

Type		Normal
Submodules	`tide`	externally forced tidal flow. Forced by input parameters `A0` and `phase0`.
Input	`L`	*Number.* Length of the system in the x-direction.
	`B`	*General 1-dimensional.* Width of the system.
	`H`	*General 1-dimensional.* Depth of the system between the reference level (i.e. water level at the mouth, typically mean sa level) and the bed.
	`Av`	*General 3-dimensional.* Vertical eddy viscosity in m^2/s.
	`roughness`	*General 3-dimensional. Second dimension is length 1.* Roughness coefficient s_f (if `BottomBC=='PartialSlip'`) or z_0 (if `BottomBC=='NoSlip'`). May vary x and time, but not in z. Therefore the second dimension needs to have length 1.
	`grid`	*iFlow grid.*
	`OMEGA`	*Number.* Angular frequency of the lowest-frequency component in rad/s
	`G`	*Number.* Acceleration of gravity in m^2/s
	`TOLERANCEBVP`	*Number.* If the bvp_solver is used to solve the ODE for the water level, this number sets the tolerance or accuracy of the result.
Input submodules	`A0`	Only `tide` *space-separated numbers.* Water level amplitude at the seaward boundary in metres. The first value corresponds to subtidal (should equal 0) and the second value corresponds to the frequency with angular frequency ω (standard M_2 tide). Unlike the `numerical2DV` package, it is not possible to include water level amplitudes higher than than the M_2 component as a forcing of the leading order hydrodynamics.
	`phase0`	Only `tide` *space-separated numbers.* Water level phase at the seaward boundary in degrees. Similar to `A0`. First element should equal 0.
Output	`zeta0`	*Numerical function 3-dimensional. Second dimension is length 1.* Leading-order water level elevation in metres. Saved as numerical function with its x- and xx-derivative.
	`u0`	*Numerical function 3-dimensional.* Horizontal flow velocity, saved as numerical function with its x-, z-, zz-, zzx-derivative.

	`w0`	*Array 3-dimensional.* Vertical flow velocity, saved as numerical function with its z-derivative.

1.1.2 HydroFirst

First-order hydrodynamics using a numerical perturbation model. See Part I of this manual.

Type		Normal
Submodules	`tide`	externally forced tidal flow. Forced by input parameters `A1` and `phase1`.
	`river`	externally forced river flow. Forced by input parameter `Q1`.
	`adv`	internally generated flow by momentum advection.
	`nostress`	internally generated flow through velocity-depth-asymmetry; interactions between the velocity gradient (i.e. the shape of the velocity profile) and the water level.
	`stokes`	internally generated tidal return flow that compensates for the net mass transport in the leading order.
	`baroc`	flow induced by a horizontal density gradient.
Input	`L`	*Number.* Length of the system in the x-direction.
	`B`	*General 1-dimensional.* Width of the system.
	`H`	*General 1-dimensional.* Depth of the system between the reference level (i.e. water level at the mouth, typically mean sa level) and the bed.
	`Av`	*General 3-dimensional.* Vertical eddy viscosity in m^2/s.
	`roughness`	*General 3-dimensional. Second dimension is length 1.* Roughness coefficient s_f (if `BottomBC=='PartialSlip'`) or z_0 (if `BottomBC=='NoSlip'`). May vary x and time, but not in z. Therefore the second dimension needs to have length 1.
	`grid`	*iFlow grid.*
	`OMEGA`	*Number.* Angular frequency of the lowest-frequency component in rad/s
	`G`	*Number.* Acceleration of gravity in m^2/s
	`TOLERANCEBVP`	*Number.* If the bvp_solver is used to solve the ODE for the water level, this number sets the tolerance or accuracy of the result.
	`RHO0`	*Number.* Reference density kg/m^3
	`BETA`	*Number.* Conversion parameter for salinity in $\rho = \rho_0(1+\beta s)$
Input submodules	`A1`	Only `tide` *space-separated numbers.* Water level amplitude at the seaward boundary in metres, see module HydroLead. Here, only a M_4 amplitude can be prescribed, i.e. the first two numbers equal 0.
	`phase1`	Only `tide` *space-separated numbers.* Water level phase at the seaward boundary in degrees. Similar to `A1`. First two elements should equal 0.

	`Q1`	Only `river` *number*. First-order river discharge at the landward boundary in m^3/s.
	`u0`	Only `stokes, nostress, adv` *General 3-dimensional* Leading-order horizontal flow velocity (m/s).
	`zeta0`	Only `stokes, nostress` *General 3-dimensional* Leading-order water level elevation (m). Second dimension should be length 1.
	`w0`	Only `adv` *General 3-dimensional* Leading-order vertical flow velocity (m/s).
	`s0`	Only `baroc` *General 3-dimensional* Leading-order salinity (psu).
Output	`zeta1`	*Numerical function 3-dimensional. Second dimension is length 1.* Leading-order water level elevation in metres. Saved as numerical function with its x- and xx-derivative.
	`u1`	*Numerical function 3-dimensional.* Horizontal flow velocity.
	`w1`	*Array 3-dimensional.* Vertical flow velocity.

1.2 Sediment

1.2.1 SedDynamic

Type		Normal
Submodules	`erosion`	internally generated sediment concentration and transport due to bottom erosion.
	`noflux`	internally generated sediment concentration and transport due to the no flux boundary condition at the surface.
	`sedadv`	internally generated sediment concentration and transport due to sediment advection also known as spatial settling lag, i.e. $uc_x + wc_z$.
Input	`L`	*Number.* Length of the system in the x-direction.
	`B`	*General 1-dimensional.* Width of the system.
	`H`	*General 1-dimensional.* Depth of the system between the reference level (i.e. water level at the mouth, typically mean sa level) and the bed.
	`Av`	*General 3-dimensional.* Vertical eddy viscosity in m^2/s.
	`roughness`	*General 3-dimensional. Second dimension is length 1.* Roughness coefficient s_f (if `BottomBC=='PartialSlip'`) or z_0 (if `BottomBC=='NoSlip'`). May vary x and time, but not in z. Therefore the second dimension needs to have length 1.
	`astar`	*Number.* Average amount of sediment at the bottom for resuspension
	`ws`	*Number.* Settling velocity in m/s
	`Kh`	*Number.* Horizontal eddy diffusivity coefficient m^2/s
	`grid`	*iFlow grid.*

	zeta0	*Numerical function 3-dimensional*. Leading-order water level elevation (m). Second dimension should be length 1.
	OMEGA	*Number*. Angular frequency of the lowest-frequency component in rad/s
	RHOS	*Number*. Density of the sea kg/m^3
	RHO0	*Number*. Reference density kg/m^3
	G	*Number*. Acceleration of gravity in m^2/s
	DS	*Number*. Sediment grain size in m
Input sub-modules	u0	all *Numerical function 3-dimensional* Leading-order horizontal flow velocity (m/s).
	zeta0	Only noflux *Numerical function 3-dimensional* Leading-order water level elevation (m). Second dimension should be length 1.
	w0	Only sedadv *Numerical function 3-dimensional* Leading-order vertical flow velocity (m/s).
	u1	Only erosion *Numerical function 3-dimensional* First-order horizontal flow velocity (m/s).
Output	hatc0	*General 3-dimensional*. Leading-order sediment concentration amplitude.
	hatc1	*General 3-dimensional*. First-order sediment concentration amplitude.
	hatc2	*General 3-dimensional*. Second-order sediment concentration amplitude. This amplitude is only due to river-river interaction.
	c0	*General 3-dimensional*. Leading-order sediment concentration.
	c1	*General 3-dimensional*. First-order sediment concentration.
	c2	*General 3-dimensional*. Second-order sediment concentration.
	T	*General 1-dimensional*. Transport function for the availability.
	F	*General 1-dimensional*. Diffusion function for the availability.
	a	*General 1-dimensional*. Availability of sediment.

2. Introduction: domain and approach

Insight into the hydrodynamical mechanisms that govern the flow and sediment transport in estuaries is essential to learn more about processes that govern the current state or the future fate of the estuary under investigation. This manual presents a detailed derivation and description of a two-dimensional semi-analytical package for iFlow that aims at this. This manual contains two parts discussing:

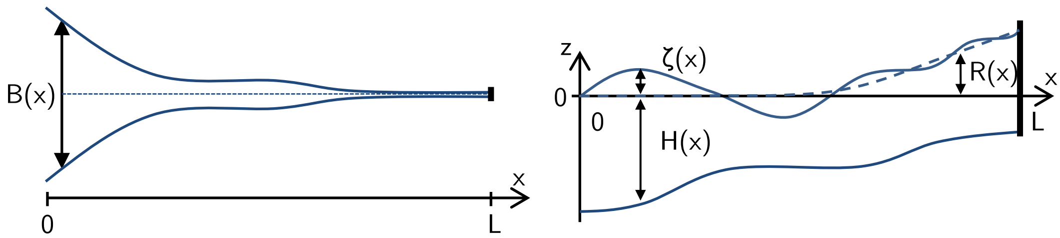
1. Hydrodynamics
2. Sediment dynamics

Every part of this manual will contain one or more chapters discussing the model equations, their derivation or solution method. The final chapter in each part contains a detailed description on the use of the provided iFlow modules.

The model is of the exploratory type (Murray, 2003) and is based on the perturbation approach, earlier adopted by e.g. Ianniello (1977, 1979); Chernetsky et al. (2010) for hydrodynamics, Chernetsky et al. (2010) for salinity and Chernetsky et al. (2010) for sediment dynamics. The perturbation approach involves a scaling of the equations to distinguish between the terms that balance at leading order and much smaller terms that balance at higher orders. Under suitable assumptions, the leading-order balance becomes linear and therefore much easier to solve than the original non-linear set of equations. The approach does however not neglect the non-linear terms or other higher-order terms. Instead, linear estimates of these terms appear as forcing mechanisms to linear higher-order balances. Theoretically, the full solution to the non-linear system is obtained when an infinite number of higher-order balances is solved for. Practically, we typically solve for the leading- and first-order balances, which provide a reasonably accurate estimate of the full solution. Due to the linearity of the equations at each order, the effect of different forcing mechanisms can be identified.

The model describes two-dimensional, width-averaged (2DV) physical quantities, i.e. surface elevation $\zeta(x,t)$, horizontal and vertical flow velocity, $u(x,z,t)$ and $w(x,z,t)$, respectively, and sediment concentration $c(x,z,t)$, in a straight channel of length L with varying longitudinal bed profile $H(x)$ and channel width $B(x)$ (Fig. 2.1). The bottom profile $H(x)$ (relative to the mean sea level defined at $z = 0$) and channel width $B(x)$ are allowed to vary gradually over the x-direction, i.e. with length scales corresponding to the length of the tidal wave. The sidewalls of the channel are assumed to be vertical and tidal flats are not present. The seaward boundary of the estuary is located at $x = 0$ and the landward boundary is located at $x = L$. At the latter boundary the river flows into the domain. Details on the functions that iFlow supports for the depth and width are provided in the manual on the auxiliary module package.

The surface level relative to $z = 0$ is expressed as $R+\zeta$ and is computed by the model. By default the reference level $R = 0$ and ζ is equal to the surface level. The use of a non-zero reference level is however required if the river bed is above MSL over part of the domain. The depth H is then negative, which poses a problem in further calculations. In this case iFlow computes the reference level R as a quick estimate of the mean surface level and ensures that $H+R$ is always positive. More details on the computation of R are provided in the part on hydrodynamics of the Numerical2DV package (Dijkstra, 2017a). In this manual, the default reference level $R = 0$ is used in the analysis.



(a) Top view (b) Side view

Figure 2.1: Model domain. The model is two-dimensional in along-channel (x) and vertical (z) direction and is width-averaged. The depth and width are allowed to vary smoothly with x.

Leading-order and first-order hydrodynamics

3 Equations and ordering **15**
3.1 Equations and assumptions
3.2 Scaling
3.3 Ordering & overview of the equations

4 Analytical solutions to the ordered equations **25**
4.1 Leading order solution
4.2 First order solutions

5 Numerical implementation **45**
5.1 General structure of the hydrodynamics modules
5.2 Solving the boundary value problem for the water level

3. Equations and ordering

3.1 Equations and assumptions

3.1.1 Model equations

It is assumed that the water motion in the estuary is dominated by tidal forcing, with effects of river discharge being relatively small. Moreover, the effect of wind stress and wind waves on the water motion is neglected. Under the above mentioned assumptions, momentum and mass balance is expressed by the width-averaged shallow water equations:

$$u_t + uu_x + wu_z = -g\zeta_x - g\int_z^\zeta \frac{\rho_x}{\rho_0}dz + (A_v u_z)_z, \tag{3.1}$$

$$u_x + w_z + \frac{B_x}{B}u = 0. \tag{3.2}$$

In these equations, $\zeta(x,t)$ is width-averaged surface elevation, and $u(x,z,t)$ and $w(x,z,t)$ is width-averaged horizontal and vertical flow velocity, respectively. Furthermore, t is time, g is gravitational acceleration, ρ is density, ρ_0 is a constant reference density and A_v is vertical eddy viscosity coefficient. The subscripts $(.)_x$, $(.)_z$ and $(.)_t$ denote the derivative of a variable in the respective dimension. In Eq. (3.1), the left-hand side contains inertia (first term) and advection (second and third term). The exerting forces are found on the right-hand side, representing barotropic pressure gradient force, baroclinic pressure gradient force and internal frictional force, respectively.

Eqs. (3.1) and (3.2) are subject to horizontal and vertical boundary conditions. Furthermore, a decision has to be made with respect to the salinity distribution and the turbulence closure. These topics are treated below.

3.1.2 Horizontal boundary conditions

The water motion is forced by a prescribed tidal elevation on the seaside of the estuary at $x = 0$ that consists of a semi-diurnal (M_2) and a M_4 constituent

$$\zeta(0,t) = A_{M_2}\cos(\sigma t) + A_{M_4}\cos(2\sigma t - \varphi), \tag{3.3}$$

where A_{M_2} and A_{M_4} are the amplitudes of the M_2 and M_4 tidal constituents, respectively, $\varphi = \varphi_{M_4} - 2\varphi_{M_2}$ the phase difference between the M_2 and M_4 tidal constituents and σ the tidal frequency.

At the landward boundary $x = L$, the channel is constrained by a weir with a constant river discharge Q. Here, the tidal discharge is required to vanish, so

$$B(L)\int_{-H(L)}^{\zeta(L,t)} u(L,z,t)dz = -Q. \tag{3.4}$$

3.1.3 Vertical boundary conditions

At the free surface $z = \zeta$, the boundary conditions are the no stress condition and the kinematic boundary condition

$$A_v u_z(x,\zeta,t) = 0, \qquad \text{(no stress)} \tag{3.5}$$
$$w(x,\zeta,t) = \zeta_t(x,t) + u(x,\zeta,t)\zeta_x(x,t). \qquad \text{(kinematic)} \tag{3.6}$$

At the bottom $z = -H$, we assume the bed to be impermeable and prescribe a partial slip condition (Schramkowski et al., 2002b)

$$w(x,-H,t) = -u(x,-H,t)H_x, \qquad \text{(impermeable bed)} \tag{3.7}$$
$$A_v u_z(x,-H,t) = s_f u(x,-H,t). \qquad \text{(partial slip)} \tag{3.8}$$

Note that the partial slip condition is evaluated at the top of the constant stress layer instead of at the true bed. Here, the parameter s_f is the so-called stress parameter that can still depend on the longitudinal coordinate. Following Friedrichs and Hamrick (1996) and Schramkowski et al. (2002a), this dependency is taken to be linear in the local water depth

$$s_f = s_{f_0}\left(\frac{H(x)}{H_0}\right)^n, \tag{3.9}$$

where s_{f_0} is constant and n a factor that generally varies between 1 and 3/2. Note that the bottom boundary condition is a linear relation between bed shear stress and velocity, which is a result of a Lorentz linearization procedure (Zimmerman, 1982).

3.1.4 Salinity distribution

The channel density $\rho(x,z,t)$ of the estuarine water varies in general due to the salinity s, suspended sediment concentration c and temperature T. In this semi-analytical package of iFlow v2.2, we neglect density gradients caused by suspended sediment and temperature and, thus, density only varies due to salinity. The equation of state for the channel density is taken to be linear (Chernetsky et al., 2010), and is given by

$$\rho = \rho_0(1+\beta_s s), \tag{3.10}$$

where s is salinity in psu and β_s a constant that converts salt to density. It is assumed that the salinity is vertically well-mixed, which means that the vertical variations of the salinity field are small compared to its depth- and time-averaged value. Hence, we can write $s \simeq \langle s(x)\rangle$, where angular brackets $\langle . \rangle$ denote a tidal average. In this report, the hydrodynamic model is diagnostic in salinity (hence density), where the expression for longitudinal salinity profile is given by (Warner et al., 2005; Talke et al., 2009)

$$s(x) = \frac{\hat{s}}{2}\left[1-\tanh\left(\frac{x-x_c}{x_L}\right)\right], \tag{3.11}$$

where $\hat{s}$ is the salinity at sea in psu, x_c is the location in the estuary in meters where the salinity gradient is largest and x_L is the length scale in meters over which the salinity decays (a measure for the size of the salt wedge).

Under the above mentioned assumptions, the baroclinic pressure gradient term in Eq. (3.1) can be rewritten as

$$g\int_z^\zeta \frac{\rho_x}{\rho_0}dz \simeq -g\beta_s\langle s_x\rangle(z-\zeta). \tag{3.12}$$

3.1.5 Turbulence closure

Following Friedrichs and Hamrick (1996), the vertical eddy viscosity coefficient A_v is parameterized as

$$A_v(x) = A_{v_0}\left(\frac{H(x)}{H_0}\right)^m, \tag{3.13}$$

where A_{v_0} is constant, m is a factor that generally varies between 0 and 3/2, and H_0 is the water depth at the entrance of the estuary. Hence, it is assumed that A_v is independent of height z and can be taken out of the parentheses in Eq. (3.1). Furthermore, asymmetry in mixing that is a result of time-dependent stratification (Stacey et al., 2001, 2010; Cheng et al., 2010) is neglected.

3.2 Scaling

The two equations for continuity (including the rewritten baroclinic pressure gradient term, Eq. (3.12)) and momentum conservation, Eqs. (3.1) and (3.2), are supplemented by the depth-averaged continuity, using Eqs. (3.2), (3.6) and (3.7). The latter is used to derive the ordinary differential equation for the water level. The three equations are repeated below

$$u_t + uu_x + wu_z = -g\zeta_x + g\beta_s\langle s_x\rangle(z-\zeta) + (A_v u_z)_z, \tag{3.14}$$

$$u_x + w_z + \frac{B_x}{B}u = 0, \tag{3.15}$$

$$\zeta_t + \left(\frac{d}{dx} + \frac{B_x}{B}\right)\int_{-H}^{\zeta} u dz = 0. \tag{3.16}$$

The equations are transformed to a dimensionless system by using a scaling argument in order to establish the order of magnitude of the several terms. The equations are scaled by using six typical scales, which are presented in Table 3.1.

Scale		Dimensionless quantity
σ	M_2 tidal frequency	$t = \sigma^{-1}\tilde{t}$
A_{M_2}	M_2 tidal amplitude at the seaward side	$\zeta = A_{M_2}\tilde{\zeta}$
L	Estuary length	$x = L\tilde{x}$
H_0	Average depth at seaward side	$z = H_0\tilde{z}$ and $H = H_0\tilde{H}$
B_0	Average width at seaward side	$B = B_0\tilde{B}$
$\mathscr{S}_x$	Typical salinity gradient	$s_x = \mathscr{S}_x\tilde{s}_{\tilde{x}}$
Derived scale		**Dimensionless quantity**
$U = \frac{\sigma A_{M_2} L}{H_0}$	Typical horizontal velocity of the M_2 tide	$u = U\tilde{u}$
$W = \frac{H_0 U}{L} = \sigma A_{\mathrm{M}_2}$	Typical vertical velocity of the M_2 tide	$w = W\tilde{w}$
$\mathscr{A}_v = \frac{\sigma H_0^2}{\lambda^2}$	Typical eddy viscosity	$A_v = \mathscr{A}_v\tilde{A}_v$

Table 3.1: Scales and derived scales for deriving the dimensionless equations.

This table presents three more scales that are derived from the other six. The velocity scale U follows from expressing the depth-averaged continuity equation in dimensionless quantities. Writing $\int_{-H}^{\zeta} udz = H\bar{u}$, where the bar denotes the depth-integrated quantity, and using the typical scales from Table 3.1 this results in

$$\sigma A_{\mathrm{M}_2}\tilde{\zeta}_{\tilde{t}} + \frac{H_0 U}{L}\left(\tilde{H}\tilde{\bar{u}}\right)_{\tilde{x}} + \frac{H_0 U}{L}\frac{\tilde{B}_{\tilde{x}}}{\tilde{B}}\tilde{H}\tilde{\bar{u}} = 0.$$

Thus, it follows that an appropriate scale U for the velocity is

$$U = \frac{\sigma A_{\mathrm{M}_2} L}{H_0}.$$

Similarly, the derived vertical velocity scale W follows from substituting the scaled variables into Eq. (3.15)

$$\frac{U}{L}\tilde{u}_{\tilde{x}} + \frac{W}{H_0}\tilde{w}_{\tilde{z}} + \frac{U}{L}\frac{\tilde{B}_{\tilde{x}}}{\tilde{B}}\tilde{u} = 0,$$

and thus

$$W = \frac{H_0 U}{L} \cdot \max\left(1, \frac{\tilde{B}_x}{\tilde{B}}\right).$$

The typical scale for the eddy viscosity follows from the stationary barotropic momentum balance $(A_v u_z)_z = g\zeta_x$. It follows that

$$\mathscr{A}_v = \frac{\sigma H_0^2}{\lambda^2},$$

where

$$\lambda = \frac{L}{L_w} = \frac{\sigma L}{\sqrt{gH_0}}.$$

Here, λ is the ratio of the estuary length L and the frictionless tidal wave length L_w up to a factor 2π.

3.2.1 Scaling the momentum equation

The dimensionless momentum equation is then given by

$$\sigma U \tilde{u}_{\tilde{t}} + \frac{U^2}{L}\tilde{u}\tilde{u}_{\tilde{x}} + \frac{WU}{H_0}\tilde{w}\tilde{u}_{\tilde{z}} = -\frac{gA_{\mathrm{M}_2}}{L}\tilde{\zeta}_{\tilde{x}} + g\beta_s\mathscr{S}_x\langle\tilde{s}_{\tilde{x}}\rangle(H_0\tilde{z} - A_{\mathrm{M}_2}\tilde{\zeta}) + \frac{\mathscr{A}_v U}{H_0^2}(\tilde{A}_v\tilde{u}_{\tilde{z}})_{\tilde{z}}.$$

Rewriting this equation yields

$$\tilde{u}_{\tilde{t}} + \frac{A_{\mathrm{M}_2}}{H_0}[\tilde{u}\tilde{u}_{\tilde{x}} + \tilde{w}\tilde{u}_{\tilde{z}}] = -\frac{1}{\lambda^2}\tilde{\zeta}_{\tilde{x}} + \mu\langle\tilde{s}_{\tilde{x}}\rangle(\tilde{z} - \frac{A_{\mathrm{M}_2}}{H_0}\tilde{\zeta}) + \frac{1}{\lambda^2}(\tilde{A}_v\tilde{u}_{\tilde{z}})_{\tilde{z}},$$

where

$$\mu = \frac{gH_0}{U}\frac{\beta_s\mathscr{S}_x}{\sigma}.$$

Here, μ is a factor determining the magnitude of the salinity gradient. The factor A_{M_2}/H_0 in front of the advection term is assumed to be much smaller than unity. This provides the motivation for ordering the equation around a small parameter ε which is defined as

$$\varepsilon = \frac{A_{M_2}}{H_0}.$$

The other factors that appear in the dimensionless momentum equation can be related to the magnitude of ε. These factors are considered below. Firstly, the magnitude of $1/\lambda^2$

depends on the ratio of the length of the system and the frictionless tidal wave and is usually considered to be close to unity. For many estuaries this is a good approximation, e.g. for the Scheldt Estuary $\lambda \approx 1.8$.

Secondly, it is assumed that the factor μ in front of the salinity gradient is of order ε. From observations in well-mixed estuaries it usually follows that currents driven by the salinity gradient are small compared to tidally driven currents.

The dimensional momentum equation then has terms of the following order of magnitude:

$$\underbrace{u_t}_{\mathscr{O}(1)} + \underbrace{uu_x}_{\mathscr{O}(\varepsilon)} + \underbrace{wu_z}_{\mathscr{O}(\varepsilon)} = \underbrace{-g\zeta_x}_{\mathscr{O}(1)} + \underbrace{g\beta_s\langle s_x\rangle(z-\zeta)}_{\mathscr{O}(\varepsilon)} + \underbrace{(A_v u_z)_z}_{\mathscr{O}(1)}$$

3.2.2 Scaling the depth-averaged continuity equation

The dimensionless form of the depth-averaged momentum equation (3.16) is

$$\tilde{\zeta}_{\tilde{t}} + \left(\frac{\partial}{\partial \tilde{x}} + \frac{\tilde{B}_{\tilde{x}}}{\tilde{B}}\right) \int\limits_{-\tilde{H}}^{\varepsilon\tilde{\zeta}} \tilde{u}\, d\tilde{z} = 0.$$

All terms are of the same order, except for the integration boundary $\varepsilon\tilde{\zeta}$. The integral is therefore linearized around $\tilde{z} = 0$ by a Taylor expansion according to

$$\int_{-\tilde{H}}^{\varepsilon\tilde{\zeta}} \tilde{u}\, d\tilde{z} = \int_{-\tilde{H}}^{0} \tilde{u}(\tilde{x},0,\tilde{t})\, d\tilde{z} + \varepsilon\tilde{\zeta}\tilde{u}(\tilde{x},0,\tilde{t}) + HOT^1$$

The dimensional equation then has terms of the following order of magnitude:

$$\underbrace{\zeta_t}_{\mathscr{O}(1)} + \left(\underbrace{\frac{\partial}{\partial x}}_{\mathscr{O}(1)} + \underbrace{\frac{B_x}{B}}_{\mathscr{O}(1)} \right) \left(\underbrace{\int_{-H}^{0} u\, dz}_{\mathscr{O}(1)} + \underbrace{\zeta u(x,0,t)}_{\mathscr{O}(\varepsilon)} \right) = 0.$$

3.2.3 Scaling the boundary conditions

Horizontal boundary conditions

The dimensionless horizontal boundary condition at the entrance $\tilde{x} = 0$ reads

$$\tilde{\zeta}(0,\tilde{t}) = \cos\tilde{t} + \frac{A_{\mathrm{M}_4}}{A_{\mathrm{M}_2}} \cos(\tilde{t} - \varphi).$$

It is assumed that A_{M_4}/A_{M_2} is of order ε and thus the dimensional form has terms of the following order of magnitude

[1] The acronym HOT means 'higher-order terms'.

$$\zeta(0,t) = \underbrace{A_{M_2}\cos(\sigma t)}_{\mathscr{O}(1)} + \underbrace{A_{M_4}\cos(2\sigma t - \phi)}_{\mathscr{O}(\varepsilon)}.$$

The landward dimensionless boundary condition at $\tilde{x} = 1$ reads

$$\int_{-\tilde{H}(1)}^{\varepsilon\tilde{\zeta}(1,\tilde{t})} \tilde{u}(1,\tilde{z},\tilde{t})d\tilde{z} = -\frac{Q}{B_0 H_0 U \tilde{B}(1)}.$$

It is assumed here that the term $Q/(B_0 H_0 U)$ is of order ε. Furthermore, the upper bound of the integral $\varepsilon\tilde{\zeta}(1,\tilde{t})$ is linearized around $\tilde{z} = 0$ by using a Taylor expansion (see previous section). The resulting dimensional form has the following order of magnitude

$$\underbrace{\int_{-H}^{0} u(L,z,t)\,dz}_{\mathscr{O}(1)} + \underbrace{\zeta(L,t)u(L,0,t)}_{\mathscr{O}(\varepsilon)} + HOT = -\underbrace{\frac{Q}{B(L)}}_{\mathscr{O}(\varepsilon)}$$

Vertical boundary conditions

The momentum equation (3.14) has boundary conditions which are applied on the bed and at the surface. The dimensionless boundary conditions at the surface $\tilde{z} = \varepsilon\tilde{\zeta}$ read

$$\begin{aligned}
&\tilde{w}(\tilde{x},\varepsilon\tilde{\zeta},\tilde{t}) = \tilde{\zeta}_{\tilde{t}}(\tilde{x},\tilde{t}) + \varepsilon\tilde{u}(\tilde{x},\varepsilon\tilde{\zeta},\tilde{t})\tilde{\zeta}_{\tilde{x}}(\tilde{x},\tilde{t}),\\
&\mathscr{A}_v\tilde{A}_v\tilde{u}_{\tilde{z}}(\tilde{x},\varepsilon\tilde{\zeta},\tilde{t}) = 0.
\end{aligned}$$

Linearizing $\tilde{w}(\tilde{x},\varepsilon\tilde{\zeta},\tilde{t})$ and $\tilde{u}(\tilde{x},\varepsilon\tilde{\zeta},\tilde{t})$ around $\tilde{z} = 0$ using a Taylor expansion results in

$$\begin{aligned}
&\tilde{w}(\tilde{x},0,\tilde{t}) + \varepsilon\tilde{\zeta}\tilde{w}_{\tilde{z}}(\tilde{x},0,\tilde{t}) = \tilde{\zeta}_{\tilde{t}}(\tilde{x},\tilde{t}) + \varepsilon\left[\tilde{u}(\tilde{x},0,\tilde{t}) + \varepsilon\tilde{\zeta}\tilde{u}_{\tilde{z}}(\tilde{x},0,\tilde{t})\right]\tilde{\zeta}_{\tilde{x}}(\tilde{x},\tilde{t}),\\
&\mathscr{A}_v\tilde{A}_v\left[\tilde{u}_{\tilde{z}}(\tilde{x},0,\tilde{t}) + \varepsilon\tilde{\zeta}\tilde{u}_{\tilde{z}\tilde{z}}(\tilde{x},0,\tilde{t})\right] = 0.
\end{aligned}$$

The resulting dimensional form of the surface boundary conditions have the following order of magnitude

$$\begin{aligned}
&\underbrace{w(x,0,t)}_{\mathscr{O}(1)} = \underbrace{\zeta_t(x,t)}_{\mathscr{O}(1)} + \underbrace{u(x,0,t)\zeta_x(x,t) - \zeta(x,t)w_z(x,0,t)}_{\mathscr{O}(\varepsilon)} + HOT,\\
&\underbrace{A_v u_z(x,0,t)}_{\mathscr{O}(1)} = -\underbrace{A_v\zeta(x,t)u_{zz}(x,0,t)}_{\mathscr{O}(\varepsilon)} + HOT.
\end{aligned}$$

The dimensionless boundary conditions on the bed $\tilde{z} = -\tilde{H}$ read

$$\tilde{A}_v \tilde{u}_{\tilde{z}}(\tilde{x}, -\tilde{H}, \tilde{t}) = \frac{s_f H_0}{\mathscr{A}_v} \tilde{u}(\tilde{x}, -\tilde{H}, \tilde{t}),$$
$$\tilde{w}(\tilde{x}, -\tilde{H}, \tilde{t}) = -\tilde{u}(\tilde{x}, -\tilde{H}, \tilde{t}) \tilde{H}_{\tilde{x}}(\tilde{x}).$$

Here $s_f H_0/\mathscr{A}_v$ is the dimensionless slip parameter. If this parameter is much smaller than unity the bottom is stress free. If it is much larger than unity, the velocity at the bed vanishes and the boundary condition reduces to the no-slip condition. The terms in these boundary conditions must all be of equal order in order to obtain balanced equations, i.e. their dimensional forms have the following order of magnitude

$$\underbrace{w(x, -H, t)}_{\mathscr{O}(1)} = -\underbrace{u(x, -H, t) H_x(x)}_{\mathscr{O}(1)}, \tag{3.17}$$
$$\underbrace{A_v u_z(x, -H, t)}_{\mathscr{O}(1)} = \underbrace{s_f u(x, -H, t)}_{\mathscr{O}(1)}. \tag{3.18}$$

3.3 Ordering & overview of the equations

The solutions u, w and ζ are written as a power series of the small parameter ε

$$u = u^0 + u^1 + u^2 + \dots,$$
$$w = w^0 + w^1 + w^2 + \dots,$$
$$\zeta = \zeta^0 + \zeta^1 + \zeta^2 + \dots,$$

where u^1, w^1 and ζ^1 are assumed to be of order ε, u^2, w^2 and ζ^2 are of order ε^2, etcetera.

Substituting these series in the momentum, continuity and depth-averaged continuity equations yields the systems of equations in leading order and first order. The solution to the momentum equation yields u, the continuity yields w and the depth-averaged continuity equation yields ζ.

3.3.1 Leading order system

At leading order, the dimensional system of equations describing the water motion reads

$$A_v u^0_{zz} - u^0_t = g\zeta^0_x, \tag{3.19}$$
$$u^0_x + w^0_z + \frac{B_x}{B} u^0 = 0, \tag{3.20}$$
$$\zeta^0_t + \left(\frac{d}{dx} + \frac{B_x}{B}\right) \int_{-H}^{0} u^0 dz = 0. \tag{3.21}$$

Note that in the momentum equation (3.19) at leading order, the advection terms uu_x and uw_z do not reappear. Additionally, the assumption that the horizontal density gradient is small has the consequence that the baroclinic pressure is of order ε and is not present in Eq. (3.19).

The corresponding boundary conditions read

$$\zeta^0 = A_{\mathrm{M}_2}\cos(\sigma t), \qquad \text{at } x = 0, \tag{3.22}$$

$$\int_{-H}^{0} u^0 dz = 0, \qquad \text{at } x = L, \tag{3.23}$$

$$w^0 = \zeta_t^0, \qquad \text{at } z = 0, \tag{3.24}$$

$$A_v u_z^0 = 0, \qquad \text{at } z = 0, \tag{3.25}$$

$$A_v u_z^0 = s_f u^0, \qquad \text{at } z = -H, \tag{3.26}$$

$$w^0 = -u^0 H_x, \qquad \text{at } z = -H. \tag{3.27}$$

3.3.2 First order system

At first order, $\mathscr{O}(\varepsilon^1)$, the dimensional momentum and continuity equations are given by

$$A_v u_{zz}^1 - u_t^1 = g\zeta_x^1 + \xi - g\beta_s\langle s_x\rangle z, \tag{3.28}$$

$$u_x^1 + w_x^1 + \frac{B_x}{B}u^1 = 0, \tag{3.29}$$

$$\zeta_t^1 + \left(\frac{d}{dx} + \frac{B_x}{B}\right)\left(\int_{-H}^{0} u^1 dz + \gamma\right) = 0, \tag{3.30}$$

where we have introduced the following simplifying notations

$$\xi(x,z,t) = u^0(x,z,t)u_x^0(x,z,t) + w^0(x,z,t)u_z^0(x,z,t), \tag{3.31}$$

$$\gamma(x,t) = \left.\zeta^0(x,t)u^0(x,t)\right|_{z=0}, \tag{3.32}$$

$$\chi(x,t) = \left.\zeta^0(x,t)u_{zz}^0(x,t)\right|_{z=0}. \tag{3.33}$$

Here, ξ represents advection of momentum and γ is related to the tidal return flow or Stokes return flow, which is the result of a positive correlation between the zeroth order vertical and horizontal tide. Finally, χ originates from the first order contribution of the stress free boundary condition.

The boundary conditions for the system of equations at first order read

$$\zeta^1 = A_{\mathrm{M}_4}\cos(2\sigma t - \varphi), \qquad \text{at } x = 0, \tag{3.34}$$

$$\int_{-H}^{0} u^1 dz = -\frac{Q}{B} - \gamma, \qquad \text{at } x = L, \tag{3.35}$$

$$w^1 = \zeta_t^1 + u^0\zeta_x^0 - w_z^0\zeta^0, \qquad \text{at } z = 0, \tag{3.36}$$

$$A_v u_z^1 = -A_v\chi, \qquad \text{at } z = 0, \tag{3.37}$$

$$A_v u_z^1 = s_f u^1, \qquad \text{at } z = -H, \tag{3.38}$$

$$w^1 = -u^1 H_x, \qquad \text{at } z = -H. \tag{3.39}$$

As is apparent from Eq. (3.28)-(3.39), the first order velocity and water level are forced externally by an M_4 tidal component and a constant river discharge and internally by the salinity gradient, the leading order advection, and tidal return flow (Stokes return flow).

4. Analytical solutions to the ordered equations

This chapter presents the derivation of the analytical solutions of the ordered equations derived in Chapter 3. It is assumed that the solution to the equations consists of a sum of tidal components and a subtidal component. Furthermore, we assume that all tidal components are overtides of the M_2 tide. This assumption allows us to eliminate the time derivatives from the equations and obtain sets of ordinary differential equations (ODEs).

4.1 Leading order solution

Since the leading order equations are forced by a single M_2 tidal component, solutions can be written in the following exponential form

$$(u^0, w^0, \zeta^0) = \tfrac{1}{2}\left[\hat{u}^0(x,z), \hat{w}^0(x,z), \hat{\zeta}^0(x,z)\right] e^{i\sigma t} + \tfrac{1}{2}\left[\hat{u}^{0^*}(x,z), \hat{w}^{0^*}(x,z), \hat{\zeta}^{0^*}(x,z)\right] e^{-i\sigma t}. \tag{4.1}$$

Here, $\hat{u}^0$, $\hat{w}^0$, $\hat{\zeta}^0$ are the complex amplitudes of the horizontal velocity, the vertical velocity and the surface elevation, respectively. Furthermore, i is the imaginary unit and the superscript $(.)^*$ denotes the complex conjugate of that variable.

Substituting the trial solution Eq. (4.1) into the momentum and continuity equations, Eqs. (3.19)-(3.21), leads to

$$A_v \hat{u}^0_{zz} - i\sigma \hat{u}^0 = g\hat{\zeta}^0_x, \tag{4.2}$$

$$\hat{u}^0_x + \hat{w}^0_z + \frac{B_x}{B}\hat{u}^0 = 0, \tag{4.3}$$

$$i\sigma\hat{\zeta}^0 + \left(\frac{d}{dx} + \frac{B_x}{B}\right)\int_{-H}^{0} \hat{u}^0 dz = 0. \tag{4.4}$$

Notice that here we only substituted the normal form of the complex amplitudes. Their conjugates follow automatically.

4.1.1 Horizontal flow velocity

First, the momentum equation (4.2) is solved to obtain an expression for the horizontal velocity amplitude $\hat{u}^0$. This equation is subject to the no stress boundary condition (3.25) at the free surface $z=0$ and the partial slip boundary condition (3.26) at the bottom $z=-H$. Using the solution (4.1), these boundary conditions transform into

$$A_v \hat{u}_z^0(x,0) = 0, \tag{4.5}$$
$$A_v \hat{u}_z^0(x,-H) - s_f \hat{u}^0(x,-H) = 0, \tag{4.6}$$

respectively. The general solution of Eq. (4.2) can be expressed in the form

$$\hat{u}^0 = \hat{u}_c^0 + \hat{u}_p^0,$$

where the particular solution $\hat{u}_p^0$ is any specific function that satisfies the inhomogeneous equation (4.2) and the complementary solution $\hat{u}_c^0$ is a general solution of the corresponding homogeneous equation. As it is assumed here that the vertical eddy viscosity is independent of z, the homogeneous equation can be written as

$$\hat{u}_{zz}^0 - \frac{i\sigma}{A_v}\hat{u}^0 = 0,$$

and the characteristic equation is

$$r_{\mathrm{M}_2}^2 - \frac{i\sigma}{A_v} = 0.$$

Consequently, the complementary solution $\hat{u}_c^0$ reads

$$\hat{u}_c^0 = C_1 e^{r_{\mathrm{M}_2} z} + C_2 e^{-r_{\mathrm{M}_2} z}, \tag{4.7}$$

with

$$r_{\mathrm{M}_2} = \sqrt{\frac{i\sigma}{A_v}}. \tag{4.8}$$

From Eq. (4.2) it follows that $\hat{u}_p^0$ is of the form $C_3 \hat{\zeta}_x^0$ and, thus, $\hat{u}_{p,z}^0 = \hat{u}_{p,zz}^0 = 0$. Substitution in Eq. (4.2) leads to

$$-i\sigma C_3 \hat{\zeta}_x^0 = g\hat{\zeta}_x^0,$$

and thus $C_3 = -g/(i\sigma)$.

The general solution of $\hat{u}^0$ is thus

$$\hat{u}^0 = C_1 e^{r_{\mathrm{M}_2} z} + C_2 e^{-r_{\mathrm{M}_2} z} - \frac{g}{i\sigma}\hat{\zeta}_x^0. \tag{4.9}$$

To obtain the coefficients C_1 and C_2, the boundary conditions (4.5) and (4.6) are used. Substituting Eq. (4.9) into Eq. (4.5) yields

$$A_v\left(r_{\mathrm{M}_2} C_1 - r_{\mathrm{M}_2} C_2\right) = 0,$$

implying that $C_1 = C_2 = C$. Subsequently, substituting Eq. (4.9) into Eq. (4.6) and solving for C leads to

$$C = \frac{1}{2}\frac{g\hat{\zeta}_x^0}{i\sigma}\frac{s_f}{A_v r_{\mathrm{M}_2}\sinh(r_{\mathrm{M}_2} H) + s_f\cosh(r_{\mathrm{M}_2} H)}\hat{\zeta}_x^0. \tag{4.10}$$

Substituting Eq. (4.10) into the general equation for $\hat{u}^0$, Eq. (4.9), and rewriting in terms of hyperbolic functions results in the expression for the horizontal velocity amplitude $\hat{u}^0$, i.e.

$$\hat{u}^0 = \frac{g\hat{\zeta}_x^0}{i\sigma}\left(\alpha_{\mathrm{M}_2}\cosh(r_{\mathrm{M}_2} z) - 1\right), \tag{4.11}$$

with

$$\alpha_{\mathrm{M}_2}(x) = \frac{s_f}{\left(A_v r_{\mathrm{M}_2}\sinh(r_{\mathrm{M}_2} H) + s_f\cosh(r_{\mathrm{M}_2} H)\right)}. \tag{4.12}$$

4.1.2 Surface elevation

Next, the solution (4.1) along with the solution for $\hat{u}^0$ are substituted into the depth-averaged continuity equation (4.4). With the third term on the left-hand side of Eq. (4.4)

$$\begin{aligned}
\frac{B_x}{B}\int_{-H}^{0}\hat{u}^0\,dz &= \frac{B_x}{B}\left[\frac{g}{i\sigma}\hat{\zeta}_x^0\left(\frac{\alpha_{\mathrm{M}_2}}{r_{\mathrm{M}_2}}\sinh(r_{\mathrm{M}_2} z) - z\right)\right]_{-H}^{0}, \\
&= \frac{B_x}{B}\frac{g}{i\sigma}\hat{\zeta}_x^0\left(\frac{\alpha_{\mathrm{M}_2}}{r_{\mathrm{M}_2}}\sinh(r_{\mathrm{M}_2} H) - H\right),
\end{aligned}$$

and the second term on the left-hand side of Eq. (4.4)

$$
\begin{aligned}
\left(\int_{-H}^{0} \hat{u}^0\, dz\right)_x &= \left(\frac{g}{i\sigma}\hat{\zeta}^0_x\left(\frac{\alpha_{\mathrm{M}_2}}{r_{\mathrm{M}_2}}\sinh(r_{\mathrm{M}_2}H) - H\right)\right)_x, \\
&= \frac{g}{i\sigma}\left[\hat{\zeta}^0_{xx}\left(\frac{\alpha_{\mathrm{M}_2}}{r_{\mathrm{M}_2}}\sinh(r_{\mathrm{M}_2}H) - H\right) + \right. \\
&\qquad \hat{\zeta}^0_x\left(\frac{\alpha_{\mathrm{M}_2,x}r_{\mathrm{M}_2} - \alpha_{\mathrm{M}_2}r_{\mathrm{M}_2,x}}{r^2_{\mathrm{M}_2}}\sinh(r_{\mathrm{M}_2}H) + \right. \\
&\qquad \left.\left.\frac{\alpha_{\mathrm{M}_2}}{r_{\mathrm{M}_2}}\cosh(r_{\mathrm{M}_2}H)(r_{\mathrm{M}_2}H_x + r_{\mathrm{M}_2,x}H) - H_x\right)\right].
\end{aligned}
$$

Eq. (4.4) then gives a second-order linear ordinary differential equation (ODE) as a function of $\hat{\zeta}^0$

$$
T_1\hat{\zeta}^0_{xx} + T_2\hat{\zeta}^0_x - T_3\hat{\zeta}^0 = 0, \tag{4.13}
$$

with

$$
\begin{aligned}
T_1 &= \frac{\alpha_{\mathrm{M}_2}}{r_{\mathrm{M}_2}}\sinh(r_{\mathrm{M}_2}H) - H, \\
T_2 &= \frac{B_x}{B}T_1 + H_x(\alpha_{\mathrm{M}_2}\cosh(r_{\mathrm{M}_2}H) - 1) + \frac{\alpha_{\mathrm{M}_2,x}}{r_{\mathrm{M}_2}}\sinh(r_{\mathrm{M}_2}H) + \\
&\quad \frac{\alpha_{\mathrm{M}_2}r_{\mathrm{M}_2,x}}{r^2_{\mathrm{M}_2}}(r_{\mathrm{M}_2}H\cosh(r_{\mathrm{M}_2}H) - \sinh(r_{\mathrm{M}_2}H)), \\
T_3 &= \frac{\sigma^2}{g}.
\end{aligned}
$$

Eq. (4.13) is subject to the boundary conditions at the seaward, $x = 0$, and landward side of the estuary, $x = L$,

$$
\hat{\zeta}^0(0) = A_{\mathrm{M}_2}, \tag{4.14}
$$

$$
\int_{-H(L)}^{0} \hat{u}^0(L)\, dz = 0. \tag{4.15}
$$

Eq. (4.13) generally needs to be solved numerically because of the non-constant coefficients in the ODE.

Intermezzo 4.1.1 — Special case. A special case in which an analytical solution for $\hat{\zeta}^0$ can be found is when α_{M_2}, r_{M_2}, and H are uniform in the x-direction and thus their derivatives w.r.t. x are zero. After rearranging terms, Eq. (4.13) then reduces to

$$\hat{\zeta}^0_{xx} + \frac{B_x}{B}\hat{\zeta}^0_x - \frac{\sigma^2}{g\left(\frac{\alpha_{\mathrm{M}_2}}{r_{\mathrm{M}_2}}\sinh(r_{\mathrm{M}_2}H) - H\right)}\hat{\zeta}^0 = 0. \tag{4.16}$$

Assuming $\hat{\zeta}^0 = e^{pt}$ to be a solution to Eq. (4.16), it follows that p must be a root of the characteristic equation

$$p^2 + \frac{B_x}{B}p - \frac{\sigma^2}{g\left(\frac{\alpha_{\mathrm{M}_2}}{r_{\mathrm{M}_2}}\sinh(r_{\mathrm{M}_2}H) - H\right)} = 0, \tag{4.17}$$

which has solutions

$$p_{1,2} = \frac{-\frac{B_x}{B} \pm \sqrt{\left(\frac{B_x}{B}\right)^2 + \frac{4\sigma^2}{g\left(\frac{\alpha_{\mathrm{M}_2}}{r_{\mathrm{M}_2}}\sinh(r_{\mathrm{M}_2}H) - H\right)}}}{2}. \tag{4.18}$$

The general solution of $\hat{\zeta}^0$ thus reads

$$\hat{\zeta}^0 = c_1 e^{p_1 x} + c_2 e^{p_2 x}. \tag{4.19}$$

Using the boundary condition at the seaward side of the estuary, Eq. (4.14), it follows that $c_1 + c_2 = A_{\mathrm{M}_2}$. From the landward boundary condition, Eq. (4.15), it follows that $\hat{\zeta}^0_x(L) = 0$. Setting the derivative of Eq. (4.19) equal to zero, leads to the following expressions for c_1 and c_2:

$$\begin{aligned} c_1 &= -\frac{A_{\mathrm{M}_2} p_2 e^{p_2 L}}{p_1 e^{p_1 L} - p_2 e^{p_2 L}}, \\ c_2 &= \frac{A_{\mathrm{M}_2} p_1 e^{p_1 L}}{p_1 e^{p_1 L} - p_2 e^{p_2 L}}. \end{aligned}$$

For readability Eq. (4.19), with the expressions for c_1 and c_2 defined above, can be rewritten in hyperbolic form. Introducing

$$p_{1,2} = -\beta \pm \beta\Gamma,$$

with

$$\beta = \frac{1}{2}\frac{B_x}{B},$$
$$\Gamma = \sqrt{1+\frac{\sigma^2}{g\beta^2\left(\frac{\alpha_{\mathrm{M}_2}}{r_{\mathrm{M}_2}}\sinh(r_{\mathrm{M}_2}H)-H\right)}},$$

the hyperbolic expression for the surface elevation $\hat{\zeta}^0$ of a converging channel with constant coefficients is

$$\hat{\zeta}^0 = \left[\frac{\Gamma\cosh\left(\beta\Gamma(x-L)\right)+\sinh\left(\beta\Gamma(x-L)\right)}{\Gamma\cosh\left(\beta\Gamma L\right)-\sinh\left(\beta\Gamma L\right)}\right]A_{\mathrm{M}_2}e^{-\beta x}. \tag{4.20}$$

4.1.3 Vertical flow velocity

Finally, the vertical velocity $\hat{w}^0$ can be found by substituting the expression for $\hat{u}^0$ into the continuity equation (4.3). This results in an expression for $\hat{w}^0_z$,

$$\begin{aligned}\hat{w}^0_z = \frac{g}{i\sigma}\Bigg[&\left(\hat{\zeta}_{xx}+\frac{B_x}{B}\hat{\zeta}_x\right)(1-\alpha_{\mathrm{M}_2}\cosh(r_{\mathrm{M}_2}z))-\\ &\hat{\zeta}_x\left(\alpha_{\mathrm{M}_2,x}\cosh(r_{\mathrm{M}_2}z)+\alpha_{\mathrm{M}_2}r_{\mathrm{M}_2,x}z\sinh(r_{\mathrm{M}_2}z)\right)\Bigg].\end{aligned} \tag{4.21}$$

Integrating Eq. (4.21) and using the kinematic boundary condition at the surface (Eq. (3.24)), $\hat{w}^0(x,0) = i\sigma\hat{\zeta}^0$, as integration constant, results in

$$\begin{aligned}\hat{w}^0 = \frac{g}{i\sigma}\Bigg[&\left(\hat{\zeta}_{xx}+\frac{B_x}{B}\hat{\zeta}_x\right)\left(z-\frac{\alpha_{\mathrm{M}_2}}{r_{\mathrm{M}_2}}\sinh(r_{\mathrm{M}_2}z)\right)-\\ &\frac{\hat{\zeta}_x}{r_{\mathrm{M}_2}}\left(\alpha_{\mathrm{M}_2,x}\sinh(r_{\mathrm{M}_2}z)+\alpha_{\mathrm{M}_2}r_{\mathrm{M}_2,x}\left(z\cosh(r_{\mathrm{M}_2}z)-\frac{\sinh(r_{\mathrm{M}_2}z)}{r_{\mathrm{M}_2}}\right)\right)-\frac{\sigma^2}{g}\hat{\zeta}^0\Bigg].\end{aligned} \tag{4.22}$$

4.2 First order solutions

As is apparent from the first order system described by Eqs. (3.28)-(3.39), the first order velocity and water level are forced externally by an M_4 tidal component and a constant river discharge and internally by the salinity gradient, the leading order advection, and tidal return flow (stokes flow). Since we assumed leading order solutions with a M_2 tidal frequency, Eq. (4.1), the product of two leading order forcing terms, Eqs. (3.31)-(3.33), with an M_2 frequency result in a residual (M_0) and a M_4 frequency. Taking $\gamma(x,t) = \zeta^0(x,t)u^0(x,0,t)$ as an example, and using Eq. (4.1), this results in

$$
\begin{aligned}
\gamma(x,t) &= \frac{1}{4}\left[\hat{\zeta}^0(x)e^{i\sigma t} + \hat{\zeta}^{0^*}(x)e^{-i\sigma t}\right]\left[\hat{u}^0(x,0)e^{i\sigma t} + \hat{u}^{0^*}(x,0)e^{-i\sigma t}\right], \\
&= \underbrace{\frac{1}{4}\left[\hat{\zeta}^0(x)\hat{u}^{0^*}(x,0) + \hat{\zeta}^{0^*}(x)\hat{u}^0(x,0)\right]}_{\mathrm{M}_0} + \underbrace{\frac{1}{4}\left[\hat{\zeta}^0(x)\hat{u}^0(x,0)e^{2i\sigma t} + \hat{\zeta}^{0^*}(x)\hat{u}^{0^*}(x,0)e^{-2i\sigma t}\right]}_{\mathrm{M}_4}, \\
&= \langle\gamma\rangle + [\gamma].
\end{aligned}
$$

In the following sections we will derive the solutions for the residual (M_0) and M_4 contributions of the surface elevation and the horizontal velocity. Thereby, we will use $\langle . \rangle$ and $[.]$ to denote the residual (M_0) or time-averaged and M_4 contribution, respectively.

4.2.1 Contributions to the residual flow velocity and surface elevation

The equations for the residual flow are obtained by taking the tide-averaged component of the first order equations. The momentum equation with its boundary conditions is then given by

$$A_v\hat{u}^{10}_{zz} = g\hat{\zeta}^{10}_x + \langle\hat{\xi}\rangle - g\beta_s\langle\hat{s}_x\rangle z, \tag{4.23}$$

$$\hat{\zeta}^{10}(0) = 0, \tag{4.24}$$

$$\int_{-H}^{0} \hat{u}^{10}(L,z)\,dz = -\frac{Q}{B(L)} - \langle\hat{\gamma}(L)\rangle, \tag{4.25}$$

$$\hat{u}^{10}_z(x,0) = -\langle\hat{\chi}\rangle, \tag{4.26}$$

$$A_v\hat{u}^{10}_z(x,-H) = s_f\hat{u}^{10}(x,-H), \tag{4.27}$$

where the first number in the superscript refers to the order and the second number to the frequency. Since the solution to $\hat{u}^{10}$ is linear, it can be constructed by adding the contributions of the different forcing terms to the residual velocity, i.e.

$$\hat{u}^{10} = \hat{u}^{\mathrm{baroc}} + \hat{u}^{\text{no-stress}} + \hat{u}^{\mathrm{stokes}} + \hat{u}^{\mathrm{river}} + \hat{u}^{\mathrm{adv}}. \tag{4.28}$$

The expressions for the contribution of each forcing term to the residual velocity can be derived by taking into account only the terms in the momentum equation and appropriate boundary conditions for each contribution.

Baroclinic pressure

For the baroclinic pressure contribution, the equations become

$$A_v\hat{u}^{\mathrm{baroc}}_{zz} = g\hat{\zeta}^{\mathrm{baroc}}_x - g\beta_s\langle\hat{s}_x\rangle z, \tag{4.29}$$

$$\int_{-H}^{0} \hat{u}^{\mathrm{baroc}}(L,z)\,dz = 0, \tag{4.30}$$

$$\hat{u}^{\mathrm{baroc}}_z(x,0) = 0, \tag{4.31}$$

$$A_v\hat{u}^{\mathrm{baroc}}_z(x,-H) = s_f\hat{u}^{\mathrm{baroc}}(x,-H). \tag{4.32}$$

Integrating Eq. (4.29) twice leads to

$$\hat{u}_z^{\text{baroc}} = \frac{g}{A_v}\left[\hat{\zeta}_x^{\text{baroc}} z - \frac{1}{2}\beta_s\langle\hat{s}_x\rangle z^2\right] + C^{\text{baroc}}, \tag{4.33}$$

$$\hat{u}^{\text{baroc}} = \frac{g}{2A_v}\left[\hat{\zeta}_x^{\text{baroc}} z^2 - \frac{1}{3}\beta_s\langle\hat{s}_x\rangle z^3\right] + C^{\text{baroc}} z + D^{\text{baroc}}, \tag{4.34}$$

where $C^{\text{baroc}}(x)$ and $D^{\text{baroc}}(x)$ are integration constants to be determined using the boundary conditions. From the boundary condition at $z=0$, Eq. (4.31), it follows that $C^{\text{baroc}}=0$. Substituting Eqs. (4.33) and (4.34) into the boundary condition at $z=-H$, Eq. (4.32), and using $C^{\text{baroc}}=0$, results in the following expression for D^{baroc}:

$$D^{\text{baroc}} = -\left(\frac{H}{s_f} + \frac{H^2}{2A_v}\right) g\zeta_x^{\text{baroc}} - \left(\frac{H^2}{2s_f} + \frac{H^3}{6A_v}\right) g\beta_s\langle\hat{s}_x\rangle. \tag{4.35}$$

Substituting $C^{\text{baroc}}=0$ and D^{baroc}, Eq. (4.35), into Eq. (4.34) results in

$$\hat{u}^{\text{baroc}} = \left(\frac{z^2-H^2}{2A_v} - \frac{H}{s_f}\right) g\zeta_x^{\text{baroc}} - \left(\frac{z^3+H^3}{6A_v} + \frac{H^2}{2s_f}\right) g\beta_s\langle\hat{s}_x\rangle. \tag{4.36}$$

The expression for $\hat{\zeta}_x^{\text{baroc}}$ is obtained by substituting Eq. (4.36) into the boundary condition at $x=1$, Eq. (4.30). Integrating Eq. (4.36) and equaling to zero leads to

$$\zeta_x^{\text{baroc}} = -\frac{\left(\frac{H}{8A_v} + \frac{1}{2s_f}\right) H\beta_s\langle\hat{s}_x\rangle}{\left(\frac{H}{3A_v} + \frac{1}{s_f}\right)}. \tag{4.37}$$

This flow contribution is also referred to as gravitational circulation. Numerically integrating Eq. (4.37) with respect to x leads to an expression for the water level set-up $\hat{\zeta}^{\text{baroc}}$ due to the baroclinic pressure gradient and the constraint of no net transport.

Stress free boundary condition

For the contribution of the stress free boundary condition, the equations become

$$A_v\hat{u}_{zz}^{\text{no-stress}} = g\hat{\zeta}_x^{\text{no-stress}}, \tag{4.38}$$

$$\int_{-H}^{0} \hat{u}^{\text{no-stress}}(L,z)\,dz = 0, \tag{4.39}$$

$$\hat{u}_z^{\text{no-stress}}(x,0) = -\langle\hat{\chi}\rangle, \tag{4.40}$$

$$A_v\hat{u}_z^{\text{no-stress}}(x,-H) = s_f\hat{u}^{\text{no-stress}}(x,-H). \tag{4.41}$$

Integrating Eq. (4.38) twice leads to

$$\hat{u}_z^{\text{no-stress}} = \frac{g}{A_v}\hat{\zeta}_x^{\text{no-stress}} z + C^{\text{no-stress}}, \tag{4.42}$$

$$\hat{u}^{\text{no-stress}} = \frac{g}{2A_v}\hat{\zeta}_x^{\text{no-stress}} z^2 + C^{\text{no-stress}} z + D^{\text{no-stress}}, \tag{4.43}$$

where $C^{\text{no-stress}}(x)$ and $D^{\text{no-stress}}(x)$ are integration constants to be determined using the boundary conditions. From the boundary condition at $z=0$, Eq. (4.40), it follows that

$$C^{\text{no-stress}} = -\langle\hat{\chi}\rangle. \tag{4.44}$$

Substituting Eqs. (4.42) and (4.43) into the boundary condition at $z=-H$, Eq. (4.41), and using the expression for $C^{\text{no-stress}}$, yields

$$D^{\text{no-stress}} = -\left(\frac{H^2}{2A_v}+\frac{H}{s_f}\right) g\hat{\zeta}_x^{\text{no-stress}} - \left(H+\frac{A_v}{s_f}\right)\langle\hat{\chi}\rangle. \tag{4.45}$$

Substituting the expressions for $C^{\text{no-stress}}$ and $D^{\text{no-stress}}$ into Eq. (4.43) results in

$$\hat{u}^{\text{no-stress}} = \left(\frac{z^2-H^2}{2A_v}-\frac{H}{s_f}\right) g\hat{\zeta}_x^{\text{no-stress}} - \left(z+H+\frac{A_v}{s_f}\right)\langle\hat{\chi}\rangle, \tag{4.46}$$

The expression for $\hat{\zeta}_x^{\text{no-stress}}$ is obtained by substituting Eq. (4.46) into the boundary condition at $x=1$, Eq. (4.39). Integrating Eq. (4.46) and equaling to zero leads to

$$\hat{\zeta}_x^{\text{no-stress}} = -\frac{\left(\frac{H}{2}+\frac{A_v}{s_f}\right)\langle\hat{\chi}\rangle}{gH\left(\frac{H}{3A_v}+\frac{1}{s_f}\right)}. \tag{4.47}$$

Numerically integrating Eq. (4.47) with respect to x leads to an expression for the water level set-up or set-down $\hat{\zeta}^{\text{no-stress}}$ due to the stress free boundary condition.

Stokes return flow

For the contribution of the Stokes return flow, the equations become

$$A_v\hat{u}_{zz}^{\text{stokes}} = g\hat{\zeta}_x^{\text{stokes}}, \tag{4.48}$$

$$\int_{-H}^{0} \hat{u}^{\text{stokes}}(L,z)\,dz = -\langle\hat{\gamma}(L)\rangle, \tag{4.49}$$

$$\hat{u}_z^{\text{stokes}}(x,0) = 0, \tag{4.50}$$

$$A_v\hat{u}_z^{\text{stokes}}(x,-H) = s_f\hat{u}^{\text{stokes}}(x,-H). \tag{4.51}$$

Integrating Eq. (4.48) twice leads to

$$\hat{u}_z^{\text{stokes}} = \frac{g}{A_v}\hat{\zeta}_x^{\text{stokes}} z + C^{\text{stokes}}, \tag{4.52}$$

$$\hat{u}^{\text{stokes}} = \frac{g}{2A_v}\hat{\zeta}_x^{\text{stokes}} z^2 + C^{\text{stokes}} z + D^{\text{stokes}}, \tag{4.53}$$

where $C^{\text{stokes}}(x)$ and $D^{\text{stokes}}(x)$ are integration constants to be determined using the boundary conditions. From the boundary condition at $z=0$, Eq. (4.50), it follows that $C^{\text{stokes}}=0$. Substituting Eqs. (4.52) and (4.53) into the boundary condition at $z=-H$, Eq. (4.51), and using the expression for C^{stokes}, yields

$$D^{\text{stokes}} = -\left(\frac{H^2}{2A_v} + \frac{H}{s_f}\right) g\hat{\zeta}_x^{\text{stokes}}. \tag{4.54}$$

Substituting the expressions for C^{stokes} and D^{stokes} into Eq. (4.53) results in

$$\hat{u}^{\text{stokes}} = \left(\frac{z^2 - H^2}{2A_v} - \frac{H}{s_f}\right) g\hat{\zeta}_x^{\text{stokes}}. \tag{4.55}$$

The expression for $\hat{\zeta}_x^{\text{stokes}}$ can be obtained by substituting Eq. (4.55) into the boundary condition at $x=1$, Eq. (4.49). This leads to

$$\hat{\zeta}_x^{\text{stokes}} = \frac{\langle\hat{\gamma}\rangle}{gH^2\left(\dfrac{H}{3A_v} + \dfrac{1}{s_f}\right)}. \tag{4.56}$$

Numerically integrating Eq. (4.56) with respect to x leads to an expression for the water level set-up $\hat{\zeta}^{\text{stokes}}$ due to the landward net transport of water induced by the tide.

River flow

For the contribution of the river flow, the equations become

$$A_v\hat{u}_{zz}^{\text{river}} = g\hat{\zeta}_x^{\text{river}}, \tag{4.57}$$

$$\int_{-H}^{0} \hat{u}^{\text{river}}(L,z)\,dz = -\frac{Q}{B(L)}, \tag{4.58}$$

$$\hat{u}_z^{\text{river}}(x,0) = 0, \tag{4.59}$$

$$A_v\hat{u}_z^{\text{river}}(x,-H) = s_f\hat{u}^{\text{river}}(x,-H). \tag{4.60}$$

Integrating Eq. (4.57) twice leads to

$$\hat{u}_z^{\text{river}} = \frac{g}{A_v}\hat{\zeta}_x^{\text{river}} z + C^{\text{river}}, \tag{4.61}$$

$$\hat{u}^{\text{river}} = \frac{g}{2A_v}\hat{\zeta}_x^{\text{river}} z^2 + C^{\text{river}} z + D^{\text{river}}, \tag{4.62}$$

where $C^{\text{river}}(x)$ and $D^{\text{river}}(x)$ are integration constants to be determined using the boundary conditions. From the boundary condition at $z = 0$, Eq. (4.59), it follows that $C^{\text{river}} = 0$. Substituting Eqs. (4.61) and (4.62) into the boundary condition at $z = -H$, Eq. (4.60), and using the expression for C^{river}, yields

$$D^{\text{river}} = -\left(\frac{H^2}{2A_v} + \frac{H}{s_f}\right) g\hat{\zeta}_x^{\text{river}}. \tag{4.63}$$

Substituting the expressions for C^{river} and D^{river} into Eq. (4.62) results in

$$u^{\text{river}} = \left(\frac{z^2 - H^2}{2A_v} - \frac{H}{s_f}\right) g\hat{\zeta}_x^{\text{river}}. \tag{4.64}$$

The expression for $\hat{\zeta}_x^{\text{river}}$ can be obtained by substituting Eq. (4.64) into the boundary condition at $x = 1$, Eq. (4.58). This leads to

$$\hat{\zeta}_x^{\text{river}} = \frac{Q}{gBH^2\left(\dfrac{H}{3A_v} + \dfrac{1}{s_f}\right)}. \tag{4.65}$$

Numerically integrating Eq. (4.65) with respect to x leads to an expression for the water level set-up $\hat{\zeta}^{\text{river}}$ due to river flow.

Advection of momentum

For the advection of momentum contribution, the equations become

$$A_v\hat{u}_{zz}^{\text{adv}} = g\hat{\zeta}_x^{\text{adv}} + \langle\hat{\xi}\rangle, \tag{4.66}$$

$$\int_{-H}^{0} \hat{u}^{\text{adv}}(L,z)\,dz = 0, \tag{4.67}$$

$$\hat{u}_z^{\text{adv}}(x,0) = 0, \tag{4.68}$$

$$A_v\hat{u}_z^{\text{adv}}(x,-H) = s_f\hat{u}^{\text{adv}}(x,-H). \tag{4.69}$$

Since the advection term $\hat{\xi}$ on the right-hand side of Eq. (4.66) is a function of z, we use the method of variation of parameters to find a solution for $\hat{u}^{\text{adv}}$. First, we must solve the homogeneous equation $\hat{u}_{zz}^{\text{adv}} = 0$, which leads to

$$\hat{u}_h^{\text{adv}} = C^{\text{adv,h}} z + D^{\text{adv,h}}, \tag{4.70}$$

where $C^{\mathrm{adv,h}}(x)$ and $D^{\mathrm{adv,h}}(x)$ are integration constants. Second, we solve the particular equation for the water level forcing, $\hat{u}_{zz}^{\mathrm{adv}} = g\hat{\zeta}_x^{\mathrm{adv}}/A_v$, which leads to

$$\hat{u}_{p,\hat{\zeta}}^{\mathrm{adv}} = \frac{g}{2A_v}\hat{\zeta}_x^{\mathrm{adv}} z^2 + C^{\mathrm{adv,p}}z + D^{\mathrm{adv,p}}, \tag{4.71}$$

where $C^{\mathrm{adv,p}}(x)$ and $D^{\mathrm{adv,p}}(x)$ are integration constants. Third, we solve the particular equation for the advective forcing, $\hat{u}_{zz}^{\mathrm{adv}} = \langle\hat{\xi}\rangle/A_v$. Since $\hat{\xi}$ is a function of z, we seek a pair of functions, $f(x,z)$ and $g(x,z)$, so that the particular solution reads

$$\hat{u}_{p,\hat{\xi}}^{\mathrm{adv}} = fz + g, \tag{4.72}$$

which has the same form as the homogeneous solution, Eq. (4.70). The first derivative of $\hat{u}_{p,\hat{\xi}}^{\mathrm{adv}}$ with respect to z is

$$(\hat{u}_{p,\hat{\xi}}^{\mathrm{adv}})_z = f_z z + f + g_z. \tag{4.73}$$

Now, let us assume that whatever f and g are, they will satisfy the following

$$f_z z + g_z = 0. \tag{4.74}$$

The first derivative $(\hat{u}_{p,\hat{\xi}}^{\mathrm{adv}})_z$ is now equal to f and the second derivative $(\hat{u}_{p,\hat{\xi}}^{\mathrm{adv}})_{zz}$ equals f_z. This implies

$$f_z = \frac{\langle\hat{\xi}\rangle}{A_v}. \tag{4.75}$$

Substituting Eq. (4.75) into Eq. (4.74) gives

$$g_z = -\frac{\langle\hat{\xi}\rangle z}{A_v}. \tag{4.76}$$

Integrating Eqs. (4.75) and (4.76) and substituting into Eq. (4.72) leads to

$$\hat{u}_{p,\hat{\xi}}^{\mathrm{adv}} = \frac{1}{A_v}\left[z\int_{-H}^{z}\langle\hat{\xi}\rangle dz' - \int_{-H}^{z}\langle\hat{\xi}\rangle z' dz'\right]. \tag{4.77}$$

Adding Eqs. (4.70), (4.71) and (4.77), results in the full expression for $\hat{u}^{\mathrm{adv}}$, viz.

$$\hat{u}^{\mathrm{adv}} = \frac{g}{2A_v}\hat{\zeta}_x^{\mathrm{adv}} z^2 + C^{\mathrm{adv}} z + D^{\mathrm{adv}} + \frac{z}{A_v}\int_{-H}^{z} \langle\hat{\xi}\rangle dz' - \frac{1}{A_v}\int_{-H}^{z} \langle\hat{\xi}\rangle z' dz', \tag{4.78}$$

where $C^{\mathrm{adv}} = C^{\mathrm{adv,h}} + C^{\mathrm{adv,p}}$ and $D^{\mathrm{adv}} = D^{\mathrm{adv,h}} + D^{\mathrm{adv,p}}$. From the boundary condition at $z = 0$, Eq. (4.68), it follows that

$$C^{\mathrm{adv}}(x) = -\frac{1}{A_v}\int_{-H}^{0} \langle\hat{\xi}\rangle dz. \tag{4.79}$$

Substituting Eq. (4.78) and Eq. (4.79) into the boundary condition at $z = -H$, Eq. (4.69), results in the following expression for D^{adv}

$$D^{\mathrm{adv}}(x) = -\left(\frac{H^2}{2A_v} + \frac{H}{s_f}\right) g\hat{\zeta}_x^{\mathrm{adv}} - \frac{1}{A_v}\left(H + \frac{A_v}{s_f}\right)\int_{-H}^{0} \langle\hat{\xi}\rangle dz. \tag{4.80}$$

Substituting the expressions for C^{adv} and D^{adv} into Eq. (4.78) results in

$$\begin{aligned} u^{\mathrm{adv}} = &\left(\frac{z^2 - H^2}{2A_v} - \frac{H}{s_f}\right) g\hat{\zeta}_x^{\mathrm{adv}} - \frac{1}{A_v}\left(z + H + \frac{A_v}{s_f}\right)\int_{-H}^{0} \langle\hat{\xi}\rangle dz \\ &+ \frac{1}{A_v}\left(z\int_{-H}^{z} \langle\hat{\xi}\rangle dz' - \int_{-H}^{z} \langle\hat{\xi}\rangle z' dz'\right). \end{aligned} \tag{4.81}$$

The expression for $\hat{\zeta}_z^{\mathrm{adv}}$ is obtained by substituting Eq. (4.81) into the boundary condition at $x = 1$, Eq. (4.67). Integrating Eq. (4.81) and equaling to zero leads to

$$\hat{\zeta}_x^{\mathrm{adv}} = \frac{\frac{1}{H}\int_{-H}^{0}\left(z\int_{-H}^{z} \langle\hat{\xi}\rangle dz' - \int_{-H}^{z} \langle\hat{\xi}\rangle z' dz'\right) dz - \left(\frac{H}{2} + \frac{A_v}{s_f}\right)\int_{-H}^{0} \langle\hat{\xi}\rangle dz}{gH\left[\frac{H}{3} + \frac{A_v}{s_f}\right]}. \tag{4.82}$$

4.2.2 Contributions to the M_4 flow velocity and surface elevation

The equations for the M_4 flow are obtained by taking the M_4-component, denoted by $[.]$, of the first order equations (3.28)-(3.38). The momentum equation, depth-averaged continuity equation and the appropriate boundary conditions are then given by

$$A_v \hat{u}^{14}_{zz} - 2i\sigma\hat{u}^{14} = g\hat{\zeta}^{14}_x + 2[\hat{\xi}], \tag{4.83}$$

$$2i\sigma\hat{\zeta}^{14} + \left(\frac{\partial}{\partial x} + \frac{B_x}{B}\right)\left(\int_{-H}^{0} \hat{u}^{14} dz + 2[\hat{\gamma}]\right) = 0, \tag{4.84}$$

$$\hat{\zeta}^{14}(0) = A_{M_4} e^{-i\varphi}, \tag{4.85}$$

$$\int_{-H}^{0} \hat{u}^{14}(L,z) dz = -2[\hat{\gamma}(L)], \tag{4.86}$$

$$\hat{u}^{14}_z(x,0) = -2[\hat{\chi}], \tag{4.87}$$

$$A_v \hat{u}^{14}_z(x,-H) = s_f \hat{u}^{14}(x,-H). \tag{4.88}$$

(4.89)

Notice, that the factor 2 appearing in the equations is due to the substitution of the trial solution Eq. (4.1). Similar to the residual velocity, the solution to $\hat{u}^{14}$ is linear and can therefore be constructed by adding the contributions of the different forcing terms, i.e.

$$\hat{u}^{14} = \hat{u}^{\text{tide}} + \hat{u}^{\text{no-stress}} + \hat{u}^{\text{stokes}} + \hat{u}^{\text{adv}} \tag{4.90}$$

The salinity field and the river outflow have no M_4 component and that an additional forcing due to the externally imposed M_4-tide is present. To derive the expressions for the different M_4 velocity contributions only the appropriate terms due to the specific forcing are taking into account. Since an additional inertia term ($2i\hat{u}^{14}$) is present in the momentum equation (4.83), solving it requires the same approach as was done for the M_2 flow velocity. In the following, the different velocity contributions are derived.

External tide

The solution for the M_4 flow velocity is derived in the same way as was done for the M_2 flow velocity in Section 4.1. The only difference is that the forcing frequency is different, i.e. $e^{2i\tilde{t}}$ instead of $e^{i\tilde{t}}$. The equations for the M_4 tide are given by

$$A_v \hat{u}^{\text{tide}}_{zz} - 2i\sigma\hat{u}^{\text{tide}} = g\hat{\zeta}^{\text{tide}}_x, \tag{4.91}$$

$$\hat{\zeta}^{\text{tide}}(0) = A_{M_4} e^{-i\varphi}, \tag{4.92}$$

$$\int_{-H}^{0} \hat{u}^{\text{tide}}(L,z) dz = 0, \tag{4.93}$$

$$\hat{u}^{\text{tide}}_z(x,0) = 0, \tag{4.94}$$

$$A_v \hat{u}^{\text{tide}}_z(x,-H) = s_f \hat{u}^{\text{tide}}(x,-H). \tag{4.95}$$

Following the same derivation steps as in Section 4.1, the solution for the M_4 flow velocity amplitude is

$$u^{\text{tide}} = \frac{g\hat{\zeta}^{\text{tide}}_x}{2i\sigma}\left(\alpha_{M_4}\cosh(r_{M_4} z) - 1\right), \tag{4.96}$$

with

$$\alpha_{M_4} = \frac{s_f}{\left(A_v r_{M_4} \sinh(r_{M_4}H) + s_f \cosh(r_{M_4}H)\right)}, \tag{4.97}$$

$$r_{M_4} = \sqrt{\frac{2i\sigma}{A_v}}. \tag{4.98}$$

To obtain the expression for $\hat{\zeta}_x^{\text{tide}}$, Eq. (4.96) is substituted in the depth-averaged continuity equation (4.84). This leads to the following second-order, linear, homogeneous ordinary differential equation (ODE) for the water level

$$F_1 \hat{\zeta}_{xx}^{\text{tide}} + F_2 \hat{\zeta}_x^{\text{tide}} - F_3 \hat{\zeta}^{\text{tide}} = 0, \tag{4.99}$$

with

$$F_1 = \frac{\alpha_{M_4}}{r_{M_4}} \sinh(r_{M_4}H) - H, \tag{4.100}$$

$$F_2 = \frac{B_x}{B} F_1 + H_x(\alpha_{M_4} \cosh(r_{M_4}H) - 1) + \frac{\alpha_{M_4,x}}{r_{M_4}} \sinh(r_{M_4}H) + \tag{4.101}$$

$$\frac{\alpha_{M_4} r_{M_4,x}}{r_{M_4}^2} (r_{M_4} H \cosh(r_{M_4}H) - \sinh(r_{M_4}H)), \tag{4.102}$$

$$F_3 = \frac{4\sigma^2}{g}, \tag{4.103}$$

Notice that Eq. (4.99) is equal to Eq. (4.13) for the M_2 water level. Furthermore, the coefficients F_1, F_2 and F_3 are similar to T_1, T_2 and T_3, respectively, except that the higher frequency of the M_4 tide is incorporated in r_{M_4}, α_{M_4} and F_3. Finally, Eq. (4.99) needs to be solved numerically since the variables α_{M_4}, r_{M_4} and H are functions of the longitudinal coordinate x.

Stress free boundary condition

The equations due to the M_4 contribution of the stress free boundary condition are given by

$$A_v \hat{u}_{zz}^{\text{no-stress}} - 2i\sigma \hat{u}^{\text{no-stress}} = g \hat{\zeta}_x^{\text{no-stress}}, \tag{4.104}$$

$$\hat{\zeta}^{\text{no-stress}}(0) = 0, \tag{4.105}$$

$$\int_{-H}^{0} \hat{u}^{\text{no-stress}}(L,z)\,dz = 0, \tag{4.106}$$

$$\hat{u}_z^{\text{no-stress}}(x,0) = -2[\hat{\chi}], \tag{4.107}$$

$$A_v \hat{u}_z^{\text{no-stress}}(x,-H) = s_f \hat{u}^{\text{no-stress}}(x,-H). \tag{4.108}$$

The general solution to the momentum equation (4.104) is given by the contribution due to the homogeneous equation and the water level forcing and reads, viz. (4.9),

$$\hat{u}^{\text{no-stress}} = C_1 e^{r_{\text{M}_4} z} + C_2 e^{-r_{\text{M}_4} z} - \frac{g}{2i\sigma}\hat{\zeta}_x^{\text{no-stress}}. \tag{4.109}$$

The unknowns C_1 and C_2 are obtained using the boundary conditions at the free surface and the bottom. From the boundary condition at the free surface, Eq. (4.107), it follows that

$$C_2 = C_1 + \frac{2[\hat{\tilde{\chi}}]}{\tilde{r}_{\text{M}_4}}. \tag{4.110}$$

Subsequently, it follows from the bottom boundary condition, Eq. (4.108), that

$$C_1 = \alpha_{\text{M}_4}\left(\frac{g\hat{\zeta}_x^{\text{no-stress}}}{4i\sigma} - \left(\frac{1}{r_{\text{M}_4}} + \frac{A_v}{s_f}\right)[\hat{\chi}]e^{r_{\text{M}_4} H}\right). \tag{4.111}$$

Substituting the expressions for C_1 and C_2, Eqs. (4.111) and (4.110), respectively, into Eq. (4.109) results in

$$\begin{aligned} u^{\text{no-stress}} =& \frac{g\hat{\zeta}_x^{\text{no-stress}}}{2i\sigma}\left(\alpha_{\text{M}_4}\cosh(r_{\text{M}_4} z) - 1\right) - \\ & \frac{2\alpha_{\text{M}_4}}{r_{\text{M}_4} s_f}[\hat{\chi}]\left(A_v r_{\text{M}_4}\cosh(r_{\text{M}_4}(z+H)) + s_f \sinh(r_{\text{M}_4}(z+H))\right). \end{aligned} \tag{4.112}$$

To obtain the expression for $\hat{\zeta}_x^{\text{no-stress}}$, Eq. (4.112) is substituted into the depth-averaged continuity equation (4.84). This leads to the following second-order, linear, non-homogeneous ODE for the water level

$$F_1\hat{\zeta}_{xx}^{\text{no-stress}} + F_2\tilde{\zeta}_x^{\text{no-stress}} - F_3\hat{\zeta}^{\text{no-stress}} = F_{\text{no-stress}}, \tag{4.113}$$

with

$$F_{\text{no-stress}} = \frac{2i\sigma}{g}\left(\left(\frac{B_x}{B}[\hat{\chi}] + [\hat{\chi}]_x\right)\frac{(1-\alpha_{\text{M}_4})}{r_{\text{M}_4}^2} - \frac{[\hat{\chi}](r_{\text{M}_4}\alpha_{\text{M}_4,x} + 2r_{\text{M}_4,x}(1-\alpha_{\text{M}_4}))}{r_{\text{M}_4}^3}\right),$$

and F_1, F_2 and F_3 defined by Eqs. (4.100)-(4.103), respectively. Eq. (4.113) needs to be solved numerically since the variables α_{M_4}, r_{M_4} and H are functions of the longitudinal coordinate x.

Stokes return flow

The equations due to the M_4 contribution of the Stokes return flow are given by

$$A_v \hat{u}^{\text{stokes}}_{zz} - 2i\sigma \hat{u}^{\text{stokes}} = g\hat{\zeta}^{\text{stokes}}_x, \tag{4.114}$$

$$\hat{\zeta}^{\text{stokes}}(0) = 0, \tag{4.115}$$

$$\int_{-H}^{0} \hat{u}^{\text{stokes}}(L,z)dz = -2[\hat{\gamma}(L)], \tag{4.116}$$

$$\hat{u}^{\text{stokes}}_z(x,0) = 0, \tag{4.117}$$

$$A_v \hat{u}^{\text{stokes}}_z = s_f \hat{u}^{\text{stokes}}. \tag{4.118}$$

The general solution to the momentum equation (4.114) is given by the contribution due to the homogeneous equation and the water level forcing. This solution is equal to Eq. (4.46) and reads

$$\hat{u}^{\text{stokes}} = C_1 e^{r_{M_4} z} + C_2 e^{-r_{M_4} z} - \frac{g}{2i\sigma}\hat{\zeta}^{\text{stokes}}_x. \tag{4.119}$$

The unknowns C_1 and C_2 are obtained using the boundary conditions at the free surface and the bottom. From the boundary condition at the free surface, Eq. (4.117), it follows that $C_1 = C_2 = C$.

Subsequently, it follows from the bottom boundary condition, Eq. (4.118), that

$$C = \frac{g\alpha_{M_4}\hat{\zeta}^{\text{stokes}}_x}{4i\sigma}. \tag{4.120}$$

Substituting the expression for C, Eq. (4.120), into Eq. (4.119) results in

$$u^{\text{stokes}} = \frac{g\hat{\zeta}^{\text{stokes}}_x}{2i\sigma}(\alpha_{M_4}\cosh(r_{M_4} z) - 1). \tag{4.121}$$

To obtain the expression for $\hat{\zeta}^{\text{stokes}}_x$, Eq. (4.121) is substituted into the depth-averaged continuity equation (4.84). This leads to the following second-order linear non-homogeneous ODE for the water level

$$F_1\hat{\zeta}^{\text{stokes}}_{xx} + F_2\tilde{\zeta}^{\text{stokes}}_x - F_3\hat{\zeta}^{\text{stokes}} = F_{\text{stokes}}, \tag{4.122}$$

with

$$F_{\text{stokes}} = -\frac{4i\sigma}{g}\left([\hat{\gamma}]_x + \frac{B_x}{B}[\hat{\gamma}]\right).$$

and F_1, F_2 and F_3 defined by Eqs. (4.100)-(4.103), respectively. Eq. (4.122) needs to be solved numerically since the variables α_{M_4}, r_{M_4} and H are functions of the longitudinal coordinate x.

Advection of momentum

The equations when the contribution of the advection of momentum is considered, become

$$A_v\tilde{u}^{\mathrm{adv}}_{zz} - 2i\sigma\hat{u}^{\mathrm{adv}} = g\hat{\zeta}^{\mathrm{adv}}_x + 2[\hat{\xi}], \tag{4.123}$$

$$\hat{\zeta}^{\mathrm{adv}}(0) = 0, \tag{4.124}$$

$$\int_{-H}^{0} \hat{u}^{\mathrm{adv}}(L,z)dz = 0, \tag{4.125}$$

$$\hat{u}^{\mathrm{adv}}_z(x,0) = 0, \tag{4.126}$$

$$A_v\hat{u}^{\mathrm{adv}}_z(x,-H) = s_f\hat{u}^{\mathrm{adv}}(x,-H). \tag{4.127}$$

Deriving the expression for $\hat{u}^{\mathrm{adv}}$, we can add the solutions to the momentum equation (4.123) due to the different forcing terms on the right-hand side of that equation. We already know the solutions of the homogeneous equation and due to the water level forcing, see Eq. (4.109). The expression for $\hat{u}^{\mathrm{adv}}$ then becomes

$$\hat{u}^{\mathrm{adv}} = C_1 e^{r_{\mathrm{M}_4}z} + C_2 e^{-r_{\mathrm{M}_4}z} - \frac{g}{2i\sigma}\hat{\zeta}^{\mathrm{adv}}_x + \hat{u}^{\mathrm{adv}}_{p,\zeta}. \tag{4.128}$$

The solution $\hat{u}^{\mathrm{adv}}_{p,\zeta}$ due to the advection of momentum is found using the method of variation of parameters. This method was already introduced in the section on the contribution of advection of momentum to the residual flow velocity. What the method boils down to is finding a solution that has a similar form as the solution to the homogeneous equation, but with different factors in front of the functions $f(x,z) = e^{r_{\mathrm{M}_4}z}$ and $g(x,z) = e^{-r_{\mathrm{M}_4}z}$. Hence, we seek functions $A(x,z)$ and $B(x,z)$, such that

$$\hat{u}^{\mathrm{adv}}_{p,\xi} = Af + Bg, \tag{4.129}$$

is a general solution of the non-homogeneous equation. We need only to calculate the integrals

$$A(x,z) = -\int_{-H}^{z} \frac{1}{W} g(x,z')b(x,z')dz', \tag{4.130}$$

$$B(x,z) = \int_{-H}^{z} \frac{1}{W} f(x,z')b(x,z')dz', \tag{4.131}$$

where W is the Wronskian of the functions f and g given by

$$W = \begin{vmatrix} e^{r_{M_4}z} & e^{-r_{M_4}z} \\ r_{M_4}e^{r_{M_4}z} & -r_{M_4}e^{-r_{M_4}z} \end{vmatrix} = -r_{M_4}e^{-r_{M_4}z}e^{r_{M_4}z} - r_{M_4}e^{-r_{M_4}z}e^{r_{M_4}z} = -2r_{M_4}.$$

Furthermore, $b(x,z) = 2[\hat{\xi}]$ is the non-homogeneous forcing term. Substituting W, f, g and b in Eqs. (4.130) and (4.131), results in

$$A(x,z) = \frac{1}{r_{M_4}} \int_{-H}^{z} [\hat{\xi}]e^{-r_{M_4}z'}dz',$$
$$B(x,z) = -\frac{1}{r_{M_4}} \int_{-H}^{z} [\hat{\xi}]e^{r_{M_4}z'}dz',$$

The final general solution to Eq. (4.66) can be found by substituting Eq. (4.129) into Eq. (4.128), using the expressions for f, g, A and B, and reads

$$\hat{u}^{\mathrm{adv}} = C_1e^{r_{M_4}z} + C_2e^{-r_{M_4}z} + \frac{1}{r_{M_4}}G_1 - \frac{g}{2i\sigma}\hat{\zeta}_x^{\mathrm{adv}}, \tag{4.132}$$

with

$$G_1 = e^{r_{M_4}z}\int_{-H}^{z} [\hat{\xi}]e^{-r_{M_4}z'}dz' - e^{-r_{M_4}z}\int_{-H}^{z} [\hat{\xi}]e^{r_{M_4}z'}dz'.$$

The unknowns C_1 and C_2 are obtained using the boundary conditions at the free surface, Eq. (4.126), and at the bottom, Eq. (4.127). At the free surface, taking the derivative of Eq. (4.132) with respect to z and equaling to zero results in

$$C_2 = C_1 + \frac{1}{r_{M_4}}G_2,$$

with

$$G_2 = \int_{-H}^{0} [\hat{\xi}]e^{-r_{M_4}z}dz + \int_{-H}^{0} [\hat{\xi}]e^{r_{M_4}z}dz.$$

At the bottom, substituting the expression for $\hat{u}^{\mathrm{adv}}$ into Eq. (4.127) and rearranging results in

$$C_1 = \alpha_{M_4}\left(\frac{g\hat{\zeta}_x^{\mathrm{adv}}}{4i\sigma} - \frac{1}{2}\left(\frac{1}{r_{M_4}} + \frac{A_v}{s_f}\right)G_2e^{r_{M_4}H}\right).$$

Substituting the expressions for C_1 and C_2 into Eq. (4.132) gives

$$\begin{aligned} u^{\mathrm{adv}} =& \frac{g\hat{\zeta}_x^{\mathrm{adv}}}{2i\sigma}(\alpha_{\mathrm{M_4}}\cosh(r_{\mathrm{M_4}}z)-1)+\frac{G_1}{A_v r_{\mathrm{M_4}}}- \\ & \frac{\alpha_{\mathrm{M_4}}G_2}{A_v r_{\mathrm{M_4}} s_f}(A_v r_{\mathrm{M_4}}\cosh(r_{\mathrm{M_4}}(z+H))+s_f\sinh(r_{\mathrm{M_4}}(z+H))). \end{aligned} \tag{4.133}$$

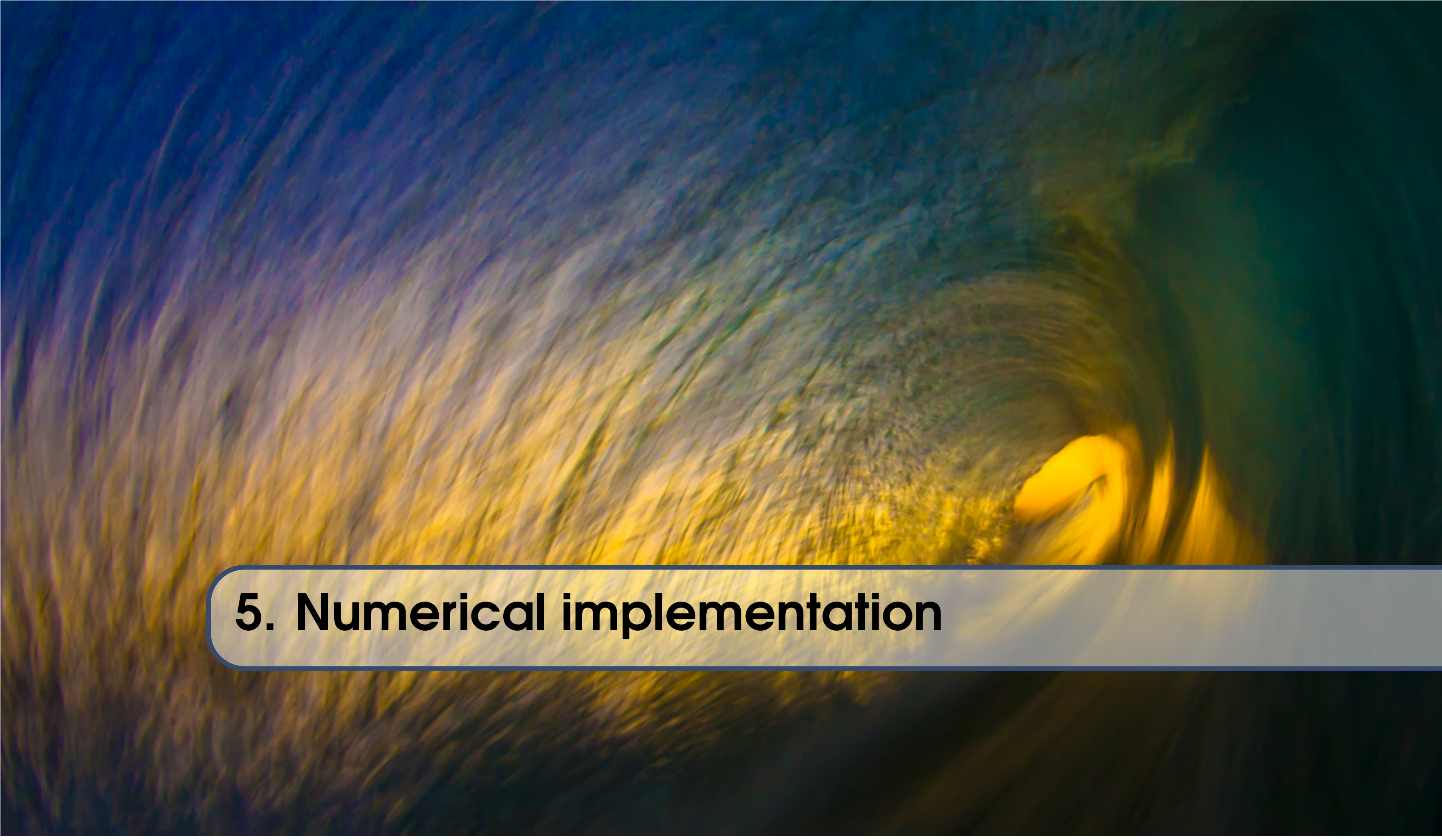
To obtain the expression for $\hat{\zeta}_x^{\mathrm{adv}}$, Eq. (4.133) is substituted into the depth-averaged continuity equation (4.84). This leads to the following second-order linear non-homogeneous ODE for the water level

$$F_1\hat{\zeta}_{xx}^{\mathrm{adv}}+F_2\hat{\zeta}_x^{\mathrm{adv}}-F_3\hat{\zeta}^{\mathrm{adv}}=F_{\mathrm{adv}}, \tag{4.134}$$

with

$$F_{\mathrm{adv}}=\frac{1}{g}\left(\frac{d}{dx}+\frac{B_x}{B}\right)\left(G_2(1-\alpha_{\mathrm{M_4}})-r\int_{-H}^{0}G_1dz\right), \tag{4.135}$$

and F_1, F_2 and F_3 defined by Eqs. (4.100)-(4.103), respectively. Eq. (4.134) needs to be solved numerically since the variables $\alpha_{\mathrm{M_4}}$, $r_{\mathrm{M_4}}$ and H are functions of the longitudinal coordinate x.



5. Numerical implementation

This chapter describes the numerical implementation of the leading and first order hydrodynamics of the semi-analytical iFlow model package discussed in Chapters 3 and 4. In the code itself, documentation is present that describes each module and function used to calculate water levels and velocities. Here, an overview of the structure of each hydrodynamics module is given, together with some more detailed information on how the equations in Chapters 3 and 4 are implemented.

5.1 General structure of the hydrodynamics modules

The leading and first order hydrodynamics are coded in two separate modules and can be found in the package folder of the used iFlow version, e.g. for version 2.4 the pathname is

```
../packages/semi_analytical2DV/hydro/
```

Both hydrodynamics modules are designed as classes in which several functions are defined; the compulsory `__init__()` and `run()` functions (see iFlow modelling framework manual) together with ones that actually calculate the surface elevation and velocities. The function structure and explanation for each module are given in Table 5.1. As can be seen from the table, both modules have similar functions and module specific functions. The functions that are similar include: calculation of the root of the characteristic equation in `rf()` and the coefficient α in `af()`, the definition of the boundary conditions in `bcs()`, and the callback functions `system_ode()` and `system_ode_der()`. The latter two are treated in more detail in the following section.

The specific functions for the `HydroLead` module are `waterlevel()` and `velocities()` in which the water level and velocities are calculated. The `HydroFirst` module contains six functions each calculating the water level and horizontal velocity due to a specific forcing mechanism. The reason is that the `HydroFirst` module loops over the first order contributions that the user defined as output in the input file.

Table 5.1: Function structure of the classes *HydroLead* and *HydroFirst*.

<table>
<tr><th>Hydrolead</th><th>HydroFirst</th></tr>
<tr><td colspan="2">- rf: Calculates the root of the characteristic equation at location x; Eqs. (4.8) and (4.98)
- af: Calculates the coefficient alpha at location x; Eqs. (4.12) and (4.97)
- bcs: Defines the boundary conditions of the boundary value problem
- system_ode: Callback function that returns the derivatives of the dependent variables in the ODE
- system_ode_der: Callback function that returns the partial derivatives of the dependent variables in the ODE</td></tr>
<tr><td>- waterlevel: Calculates the water level due to external M_2 tide by solving the boundary value problem; Eq. (4.13)
- velocities: Calculates the horizontal and vertical flow velocities; Eq. (4.11)</td><td>- tide: Calculates the water level and horizontal flow velocities due to external M_4 tide; Eqs. (4.96) and (4.99)
- stokes: Calculates the water level and horizontal flow velocities due to Stokes return flow; Eqs. (4.55) and (4.122)
- nostress: Calculates the water level and horizontal flow velocities due to the no-stress boundary condition; Eqs. (4.46) and (4.113)
- adv: Calculates the water level and horizontal flow velocities due to advection of momentum; Eqs. (4.81) and (4.134)
- baroc: Calculates the water level and horizontal flow velocities due to the baroclinic pressure gradient; Eq. (4.36)
- river: Calculates the water level and horizontal flow velocities due to river flow; Eq. (4.64)</td></tr>
</table>

5.2 Solving the boundary value problem for the water level

The two hydrodynamics modules both use the `scikits.bvp_solver` package (see https://pythonhosted.org/scikits.bvp_solver/index.html to find the documentation and how to install it) to solve the differntial equation for the water level due to the external M_2 tide, Eq. (4.13), the external M_4 tide, Eq. (4.99), the stokes return flow, Eq. (4.122), the no-stress boundary condition, Eq. (4.113), and the advection of momentum, Eq. (4.134). The general form of the second order differential equation for the water level is

$$T_1 \hat{\zeta}_{xx} + T_2 \hat{\zeta}_x - T_3 \hat{\zeta} = T_4. \tag{5.1}$$

For the numerical implementation of this differential equation in the BVP solver (found in the function `system_ode`), it needs to be transformed into two first order differential equations. Introducing $Y_1 = \hat{\zeta}$ and $Y_2 = \hat{\zeta}_x$, Eq. (5.1) can be recast to

$$\frac{dY_1}{dx} = Y_2, \tag{5.2}$$

$$\frac{dY_2}{dx} = \frac{1}{T_1}\left(T_3 Y_1 - T_2 Y_2 + T_4\right). \tag{5.3}$$

Unfortunately, `scikit.bvp_solver` cannot handle complex numbers in solving the problem. Therefore, Eqs. (5.2)-(5.3) need to be split into real and imaginary parts. Introducing $Y_1 = y_1 + iy_2$ and $Y_2 = y_3 + iy_4$, this results in the following four equations that are implemented in hydrodynamics modules of the function `system_ode`

$$
\begin{aligned}
\frac{dy_1}{dx} &= y_3, \\
\frac{dy_2}{dx} &= y_4, \\
\frac{dy_3}{dx} &= Re\left\{\tfrac{T_3}{T_1}\right\} y_1 - Im\left\{\tfrac{T_3}{T_1}\right\} y_2 - Re\left\{\tfrac{T_2}{T_1}\right\} y_3 + Im\left\{\tfrac{T_2}{T_1}\right\} y_4 + Re\left\{\tfrac{T_4}{T_1}\right\}, \\
\frac{dy_4}{dx} &= Im\left\{\tfrac{T_3}{T_1}\right\} y_1 + Re\left\{\tfrac{T_3}{T_1}\right\} y_2 - Im\left\{\tfrac{T_2}{T_1}\right\} y_3 + Re\left\{\tfrac{T_2}{T_1}\right\} y_4 + Im\left\{\tfrac{T_4}{T_1}\right\},
\end{aligned}
$$

where $Re\{.\}$ and $Im\{.\}$ denote the real and imaginary part, respectively. Note that for the external M_2 and M_4 tidal forcing, $T_4 = 0$.

To speed up computation time, `scikits.bvp_solver` allows the user to define the analytical partial derivatives of the derivatives of the dependent variables, i.e. $dy_j/dx = y_j'$ $(j = 1, \ldots, 4)$, resulting in the following matrix that is implemented in the function `system_ode_der`

$$
\begin{bmatrix}
\frac{\partial y_1'}{\partial y_1} & \cdots & \frac{\partial y_1'}{\partial y_4} \\
\vdots & \ddots & \vdots \\
\frac{\partial y_4'}{\partial y_1} & \cdots & \frac{\partial y_4'}{\partial y_4}
\end{bmatrix}
=
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
Re\left\{\tfrac{T_3}{T_1}\right\} & -Im\left\{\tfrac{T_3}{T_1}\right\} & -Re\left\{\tfrac{T_2}{T_1}\right\} & Im\left\{\tfrac{T_2}{T_1}\right\} \\
Im\left\{\tfrac{T_3}{T_1}\right\} & Re\left\{\tfrac{T_3}{T_1}\right\} & -Im\left\{\tfrac{T_2}{T_1}\right\} & -Re\left\{\tfrac{T_2}{T_1}\right\}
\end{bmatrix}.
$$

With the calculated water levels of each forcing mechanism known, the corresponding velocities are calculated using the analytical expressions derived in Chapter 4.

II

Sediment dynamics

6 Equations and ordering 51

6.1 Equations and assumptions

6.2 Scaling

6.3 Ordering & overview of the equations

7 Analytical solutions to the ordered equations 61

7.1 Leading order solutions of sediment concentration

7.2 First order solutions of sediment concentration

7.3 Second order solution of sediment concentration due to erosion by river flow

7.4 Morphodynamic equilibrium condition; transport components

8 Numerical implementation 71

8.1 General structure of the sediment dynamics module

6. Equations and ordering

6.1 Equations and assumptions

6.1.1 Sediment concentration equation

In addition to hydrodynamic model discussed in the previous part, the morphodynamic model calculates the width-averaged sediment concentration $c(x,z,t)$ in the model domain. The sediment is assumed to consist of non-cohesive, fine particles that have a uniform grain size (constant settling velocity) and are transported primarily as suspended load. The sediment dynamics is described by the width-averaged sediment mass balance equation (for a detailed derivation of this equation see Chernetsky, 2012)

$$c_t + uc_x + wc_z = w_s c_z + (K_h c_x)_x + \frac{B_x}{B} K_h c_x + (K_v c_z)_z, \tag{6.1}$$

where, $c(x,z,t)$ is the width-averaged suspended sediment concentration, w_s is settling velocity and K_h and K_v are the horizontal and vertical eddy diffusivity coefficient, respectively. Usually, K_v is assumed to be equal to the vertical eddy viscosity coefficient A_v. On the left-hand side of Eq. (6.1), the first term is associated with temporal settling lag effects (related to tidal asymmetry and local inertia, see Groen, 1967) and the second and third term with spatial settling lag effects (related to the finite time for sediment particles to settle, see Postma, 1954; de Swart and Zimmerman, 2009). On the right-hand side, the first term is associated with the settling of sediment, whereas the other terms are due to diffusive transport processes.

6.1.2 Vertical boundary conditions

At the free surface $z = \zeta$, there is no transport of sediment through the water surface

$$w_s c(x,\zeta,t) + K_v c_z(x,\zeta,t) - K_h c_x(x,\zeta,t)\zeta_x(x,t) = 0 \tag{6.2}$$

At the bottom $z = -H$, it is assumed that the diffusive flux equals the erosion flux E

$$-K_v c_z(x,-H,t)n_z - K_h c_x(x,-H,t)n_x = E. \tag{6.3}$$

Here, $\vec{n} = (n_x, n_z) = (H_x/|\vec{n}|, 1/|\vec{n}|)$ is the unit normal vector at the bottom and the erosion flux E is related to the so-called reference concentration $c_\star$ through $E = w_s c_\star$. In turn, the reference concentration is defined as

$$c_\star(x,t) = \rho_s \frac{|\tau_b(x,t)|}{\rho_0 g' d_s} a(x), \tag{6.4}$$

where, ρ_s is density of sediment, $\tau_b(x,t)$ is the bed shear stress defined as

$$\tau_b = \rho_0 A_v u_z = \rho_0 s_f u. \tag{6.5}$$

Here, ρ_0 is reference density, $g' = g(\rho_s - \rho_0)/\rho_0$ is reduced gravity, d_s is grain size, and $a(x)$ is the availability of easily erodible sediment in mud reaches. The availability function is unknown and is yet to be determined. This can be achieved by using the *morphodynamic equilibrium condition* discussed in the next section.

6.1.3 Morphodynamic equilibrium condition

Following Friedrichs et al. (1998), we assume that the total amount of sediment in the estuary varies on a timescale that is much longer than that at which the easily erodible sediment is redistributed. In that case, the availability of sediment can be determined by assuming that the tidally averaged transport of sediment is divergence free, i.e. there is a balance between the tidally averaged erosion and deposition at the bottom $z = -H(x)$. This is also known as the *morphodynamic equilibrium condition*. Assuming that there is no residual sediment flux through the seaward and landward boundaries, we can write this condition as (for a detailed derivation of this equation see Chernetsky, 2012)

$$B \left\langle \int_{-H}^{\zeta} (uc - K_h c_x) dz \right\rangle = 0. \tag{6.6}$$

The sediment concentration in the morphodynamic equilibrium still depends on the unknown availability of sediment $a(x)$. Since the sediment concentration depends linearly on the availability of sediment, the morphodynamic equilibrium condition, Eq. (6.6), can be rewritten in terms of $a(x)$.

$$F a_x + T a = 0, \tag{6.7}$$

with

$$F = \left\langle \int_{-H}^{\zeta} \frac{uc^{ax}}{a_x} - K_h \frac{c^a}{a} dz \right\rangle, \tag{6.8}$$

$$T = \left\langle \int_{-H}^{\zeta} (\frac{uc^a}{a} - K_h \left(\frac{c^a}{a}\right)_x) dz \right\rangle, \tag{6.9}$$

where c^a and c^{ax} are the parts of c that are a function of a and a_x, respectively. The solution to Eq. (6.7) is

$$a(x) = Ae^{-\int_0^x \frac{T}{F} dx}, \tag{6.10}$$

where A is an integration constant. Instead of an initial condition at the seaward boundary, we prescribe the average amount of sediment at the bottom for resuspension,

$$a_\star = \frac{\int_0^L B(x) a(x) dx}{\int_0^L B(x) dx}. \tag{6.11}$$

Substituting the Eq. (6.10) into Eq. (6.11) results in an expression for the integration constant A

$$A = \frac{a_\star \int_0^L B(x) dx}{\int_0^L B(x) e^{-\int_0^x \frac{T}{F} dx} dx}. \tag{6.12}$$

6.2 Scaling

Similar to the momentum and mass balance equations for the water motion and corresponding boundary conditions, the sediment mass balance equation and its boundary conditions are transformed to a dimensionless form to determine the order of magnitude of each term. The typical scales presented in Table 3.1 are augmented with one for the sediment concentration, C, the availability of sediment, $a_\star$ and the horizontal and vertical eddy diffusivities, see Table 6.1.

A typical scale for the sediment concentration follows by scaling the reference concentration and using the expression $\tau_b = \rho_0 A_v u_z$

$$\tilde{c}_\star = C\tilde{A}_v |\tilde{u}_{\tilde{z}}| \tilde{a}, \tag{6.13}$$

Scale		Dimensionless quantity
$C = \frac{\rho_s \mathscr{A}_v U a_\star}{H_0 g' d_s}$	Typical sediment concentration	$c = C\tilde{c}$
$a_\star$	Typical amount of available sediment	$a = a_\star \tilde{a}$
$\mathscr{K}_v = \mathscr{A}_v = \frac{\sigma H_0^2}{\lambda^2}$	Typical vertical eddy diffusivity	$K_v = \mathscr{K}_v \tilde{K}_v$
$\mathscr{K}_h = \sigma L^2 \left(\frac{A_{\mathrm{M}_2}}{H_0}\right)^2$	Typical horizontal eddy diffusivity	$K_h = \mathscr{K}_h \tilde{K}_h$

Table 6.1: Typical scales for deriving the dimensionless sediment mass balance equation. See Table 3.1 for the typical scales used for the equations for the water motion.

with

$$C = \frac{\rho_s \mathscr{A}_v U a_\star}{H_0 g' d_s}. \tag{6.14}$$

The vertical eddy diffusivity is usually taken equal to the eddy viscosity and hence their typical scale are the same.

The typical scale for the horizontal eddy diffusivity can be found by assuming that it can be represented by a velocity scale multiplied by a length scale. Here, we take the tidal velocity U and the tidal excursion length $\ell = U/\sigma$, leading to the following typical scale for the horizontal eddy diffusivity,

$$\mathscr{K}_h = U\ell = \frac{U^2}{\sigma} = \sigma L^2 \left(\frac{A_{\mathrm{M}_2}}{H_0}\right)^2. \tag{6.15}$$

6.2.1 Scaling the sediment mass balance equation

Using these typical scales, the dimensionless sediment mass balance equation is

$$\tilde{c}_{\tilde{t}} + \frac{A_{\mathrm{M}_2}}{H_0}\left[\tilde{u}\tilde{c}_{\tilde{x}} + \tilde{w}\tilde{c}_{\tilde{z}}\right] = \frac{w_s}{\sigma H_0}\tilde{c}_{\tilde{z}} + \frac{\mathscr{K}_h}{\sigma L^2}(\tilde{K}_h \tilde{c}_{\tilde{x}})_{\tilde{x}} + \frac{\mathscr{K}_h}{\sigma L^2}\frac{\tilde{B}_{\tilde{x}}}{\tilde{B}}\tilde{K}_h \tilde{c}_{\tilde{x}} + \frac{\mathscr{K}_v}{\sigma H_0^2}(\tilde{K}_v \tilde{c}_{\tilde{z}})_{\tilde{z}}. \tag{6.16}$$

Similar to the hydrodynamics, we recognize the small parameter $\varepsilon = A_{\mathrm{M}_2}/H_0$ in front of the sediment advection term. The factors in front of the other terms can be related to ε. First, from observations the factor $w_s/\sigma H_0$ in front of the first rhs term associated with the settling of sediment is usually close to unity. Second, the factor $\mathscr{K}_h/\sigma L^2$ is of order ε^2. Finally, the vertical diffusion coefficient, K_v, is usually taken equal to the vertical eddy viscosity coefficient, A_v. Hence, the factor in front of the vertical diffusion of sediment term $\mathscr{K}_v/\sigma H_0^2$ is of order one.

The dimensional sediment mass balance equation thus has terms of the following order of magnitude:

$$\underbrace{c_t}_{\mathscr{O}(1)} + \underbrace{uc_x}_{\mathscr{O}(\varepsilon)} + \underbrace{wc_z}_{\mathscr{O}(\varepsilon)} = \underbrace{w_s c_z}_{\mathscr{O}(1)} + \underbrace{(K_h c_x)_x}_{\mathscr{O}(\varepsilon^2)} + \underbrace{\frac{B_x}{B}K_h c_x}_{\mathscr{O}(\varepsilon^2)} + \underbrace{(K_v c_z)_z}_{\mathscr{O}(1)}$$

6.2.2 Scaling the vertical boundary conditions

The sediment mass balance equation (6.1) has boundary conditions that act on the bed and at the surface. The dimensionless boundary condition at the surface $\tilde{z} = \varepsilon\tilde{\zeta}$ reads

$$\frac{w_s}{\sigma H_0}\tilde{c}(\tilde{x},\tilde{\zeta},\tilde{t}) + \frac{\mathscr{K}_v}{\sigma H_0^2}\tilde{K}_v\tilde{c}_{\tilde{z}}(\tilde{x},\tilde{\zeta},\tilde{t}) - \frac{\varepsilon\mathscr{K}_h}{\sigma L^2}\tilde{K}_h\tilde{c}_{\tilde{x}}(\tilde{x},\tilde{\zeta},\tilde{t})\tilde{\zeta}_{\tilde{x}}(\tilde{x},\tilde{t}) = 0. \tag{6.17}$$

According to the typical scales introduced in Tables 3.1 and 6.1, the first and second terms are of order one, whereas the third term is of order ε^3. The resulting dimensional form of the surface boundary condition has the following order of magnitude

$$\underbrace{w_s c(x,\zeta,t)}_{\mathscr{O}(1)} + \underbrace{K_v c_z(x,\zeta,t)}_{\mathscr{O}(1)} - \underbrace{K_h c_x(x,\zeta,t)\zeta_x(x,t)}_{\mathscr{O}(\varepsilon^3)} = 0 \tag{6.18}$$

At the bottom $\tilde{z} = -\tilde{H}$, the dimensionless boundary condition is

$$-\frac{\mathscr{K}_v}{\sigma H_0^2}\tilde{K}_v\tilde{c}_{\tilde{z}}(\tilde{x},-\tilde{H},\tilde{t}) - \frac{\mathscr{K}_h}{\sigma L^2}\tilde{K}_h\tilde{c}_{\tilde{x}}(\tilde{x},-\tilde{H},\tilde{t})\tilde{H}_{\tilde{x}} = \frac{w_s}{\sigma H_0}\tilde{A}_v|\tilde{u}_{\tilde{z}}|\tilde{a}. \tag{6.19}$$

Apart from the second term on the left-hand side, which is of order ε^3, all the terms are of order one. Hence, the dimensional bottom boundary condition has the following order of magnitude

$$-\underbrace{K_v c_z(x,-H,t)n_z}_{\mathscr{O}(1)} - \underbrace{K_h c_x(x,-H,t)n_x}_{\mathscr{O}(\varepsilon^2)} = \underbrace{\frac{w_s\rho_s s_f}{g' d_s}|u(x,-H,t)|a(x)}_{\mathscr{O}(1)}. \tag{6.20}$$

Notice that in Eq. (6.19) we have used $\tau_b = \rho_0 A_v u_z$, whereas in Eq. (6.20) we have used $\tau_b = \rho_0 s_f u$. Both expressions are possible, see Eq. (6.5), and are in accordance with the partial slip boundary condition.

6.2.3 Scaling the morphodynamic equilibrium condition

The dimensionless morphodynamic equilibrium condition is

$$\left\langle \int_{-\tilde{H}}^{\varepsilon\tilde{\zeta}} (U\tilde{u}\tilde{c} - \frac{\mathscr{K}_h}{L}\tilde{K}_h\tilde{c}_{\tilde{x}})d\tilde{z} \right\rangle = 0, \tag{6.21}$$

where we have omitted the width B in this equation. By substituting the typical scales U and $\mathscr{K}_h$ in this equation, we find that the first term is an order ε term and the second term is an order ε^2 term. However, we know that the tidally averaged sediment transport

$\langle \tilde{u}\tilde{c} \rangle$ is of order ε as well, whereas the longitudinal gradient of the sediment concentration $\langle \tilde{c}_{\tilde{x}} \rangle$ is of order one. As a result, both terms in Eq. (6.21) are of the same order ε^2 and the dimensional expression for the morphodynamic equilibrium condition has the following order of magnitude

$$\left\langle \int_{-H}^{\zeta} (\underbrace{uc}_{\mathscr{O}(\varepsilon^2)} - \underbrace{K_h c_x}_{\mathscr{O}(\varepsilon^2)}) dz \right\rangle = 0. \tag{6.22}$$

Furthermore, we scale expression (6.11) that prescribes the average amount of sediment in the system, resulting in

$$\frac{\int_0^1 \tilde{B}\tilde{a} d\tilde{x}}{\int_0^1 \tilde{B} d\tilde{x}} = 1. \tag{6.23}$$

It follows, that the nominator and the denominator are of the same order.

6.3 Ordering & overview of the equations

As introduced in Section 3.2, the solution u, w, ζ and here also c are written as a power series of the small parameter ε

$$\begin{aligned} u &= u^0 + u^1 + u^2 + \dots, \\ w &= w^0 + w^1 + w^2 + \dots, \\ \zeta &= \zeta^0 + \zeta^1 + \zeta^2 + \dots, \\ c &= c^0 + c^1 + c^2 + \dots, \end{aligned}$$

where u^1, w^1 and ζ^1 are assumed to be of order ε, u^2, w^2 and ζ^2 are of order ε^2, etcetera.

Substituting these series in the sediment mass balance equation, the boundary conditions and the morphodynamic equilibrium condition yields the systems of equations in leading order and first order.

6.3.1 Leading order concentration equation

At leading order, when assuming a constant vertical eddy diffusivity, the dimensional sediment mass balance equation reads

$$c_t^0 - w_s c_z^0 - K_v c_{zz}^0 = 0, \tag{6.24}$$

with boundary conditions

$$w_s c^0(x,0,t) + K_v c_z^0(x,0,t) = 0, \tag{6.25}$$

$$-K_v c_z^0(x,-H,t) = \hat{E}^0 a(x), \tag{6.26}$$

with

$$\hat{E}^0 = \frac{w_s \rho_s}{\rho_0 g' d_s} |\tau_b^0(x,t)| = \frac{w_s \rho_s s_f}{g' d_s} |u_b^{02}(x,t)|. \tag{6.27}$$

In these equations, the subscript $[.]_b$ denotes evaluation of the physical variable at the bottom and $\hat{E}^0$ is the leading order erosion term (see Intermezzo 6.3.1). It thus follows from Eqs. (6.24)-(6.27) that the leading order sediment concentration is forced internally by the leading order bottom shear stress.

Intermezzo 6.3.1 — Derivation of the first and leading order erosion terms, $\hat{E}^0$ and $\hat{E}^1$. In this intermezzo a derivation is given for the erosion terms $\hat{E}^0$ and $\hat{E}^1$ that arise in the bottom boundary conditions of the leading and first order contributions to the sediment concentration.

Using the power series expression for u and restricting attention up to first order terms, with $u^0 \gg u^1$, we can write the magnitude of bottom shear stress τ_b as

$$\begin{aligned}
|\tau_b| &= \rho_0 s_f |u| = \rho_0 s_f \sqrt{(u^0+u^1)^2} \approx \rho_0 s_f \sqrt{(u^0)^2 + 2u^0 u^1}, \\
&= \rho_0 s_f |u^0| \sqrt{1 + 2\frac{u^1}{u^0}} \approx \rho_0 s_f |u^0| + \rho_0 s_f \frac{|u^0|}{u^0} u^1, \\
&\approx \rho_0 s_f |u^0| + \rho_0 s_f \mathrm{sg}(u^0) u^1,
\end{aligned}$$

where the first term is $\mathcal{O}(1)$ and the second $\mathcal{O}(\varepsilon)$. Hence, we find that

$$\hat{E} = \frac{w_s \rho_s}{\rho_0 g' d_s} |\tau_b| \approx \hat{E}^0 + \hat{E}^1,$$

with

$$\begin{aligned}
\hat{E}^0 &= \frac{w_s \rho_s s_f}{g' d_s} |u_b^{02}|, \\
\hat{E}^1 &= \frac{w_s \rho_s s_f}{g' d_s} \mathrm{sg}(u^{02}) \left[u_b^{10} + u_b^{14} \right].
\end{aligned}$$

To determine the leading order erosion $\hat{E}^0$ we need the harmonic decomposition of $|u_b^{02}|$. This term only contains residual components and tidal components that are even multiples of the M_2 tide (M_4 etc.). More specifically, when writing u^{02} as

$$u^{02} = \tfrac{1}{2}\hat{u}^{02}e^{i\sigma t} + c.c.,$$

the following harmonic series for $|u^{02}|$ holds,

$$|u_b^{02}| = \sum_{n=-\infty}^{\infty} a_{2n}e^{2in\sigma t},$$

with

$$a_{2n} = \frac{2}{\pi}|\hat{u}^{02}|\left(\frac{\hat{u}_b^{02}}{\hat{u}_b^{02*}}\right)^n \frac{(-1)^n}{1-4n^2}.$$

For the leading order concentration only the residual and M_4 component are relevant ($n=0$ and $n=\pm 1$). Similarly, for the first order erosion term $\hat{E}^1$ the harmonic components of sg(u^{02}) are relevant, which can be written in the following harmonic series

$$\mathrm{sg}(u_b^{02}) = \sum_{n=-\infty}^{\infty} a_{(2n+a)}e^{i(2n+1)\sigma t},$$

with

$$a_{(2n+1)} = \frac{2(-1)^n}{\pi(2n+1)}\left(\frac{\hat{u}_b^{02}}{|\hat{u}_b^{02}|}\right)^{2n+1}.$$

For the first order concentration only the M_2 and M_6 components are relevant ($n=0$ and $n=\pm 1$)

6.3.2 First order concentration equation

At first order, the dimensional sediment mass balance equation is

$$c_t^1 - w_s c_z^1 - K_v c_{zz}^1 = -u^0 c_x^0 - w^0 c_z^0. \tag{6.28}$$

The corresponding boundary conditions are

$$w_s c^1(x,0,t) + K_v c_z^1(x,0,t) = -\zeta^0(x,t)c_t^0(x,0,t), \tag{6.29}$$
$$-K_v c_z^1(x,-H,t) = \hat{E}^1 a(x), \tag{6.30}$$

with

$$\hat{E}^1 = \frac{w_s \rho_s}{\rho_0 g' d_s} |\tau_b^1(x,t)| = \frac{w_s \rho_s s_f}{g' d_s} \mathrm{sg}(u_b^{02}(x,t)) u_b^1(x,t). \tag{6.31}$$

Here, $\hat{E}^1$ is the first order erosion term (see Intermezzo 6.3.1). It follows from Eqs. (6.28)-(6.30) that the first order sediment concentration is forced internally by the first order bottom shear stress (Eq. (6.30)), advection of sediment (Eq. (6.28)) and surface correction term due to a Taylor expansion around $z = 0$ (Eq. (6.29)).

6.3.3 Second order concentration equation

In general, we do not consider the sediment balance at second order. However, near the landward boundary all tidal velocity components vanish, and thus there is hardly any tidal transport, whereas river flow might still be strong enough to erode sediment and transport it downstream. This transport mechanism is due to a second order, river flow-induced sediment concentration $c^{20}_{\text{river-river}}$ that is subsequently advected by the river flow. We argue that this transport mechanism becomes dominant over tidal transport mechanisms near the landward boundary and thus we include this mechanism in the iFlow's sediment dynamics.

The corresponding sediment mass balance equation at second order is

$$(c^{20}_{\text{river-river}})_t - w_s (c^{20}_{\text{river-river}})_z - K_v (c^{20}_{\text{river-river}})_{zz} = 0, \tag{6.32}$$

with boundary conditions

$$w_s c^{20}_{\text{river-river}}(x,0,t) + K_v (c^{20}_{\text{river-river}}(x,0,t))_z = 0, \tag{6.33}$$
$$-K_v (c^{20}_{\text{river-river}}(x,-H,t))_z = \hat{E}^{20}_{\text{river-river}} a(x), \tag{6.34}$$

with

$$\hat{E}^{20}_{\text{river-river}} = \frac{w_s \rho_s s_f}{g' d_s} \left[\langle |u_b^{02}(x,t) + \hat{u}^{10}_{b,\text{river}}(x,t)| \rangle - \langle |u_b^{02}(x,t)| \rangle \right]. \tag{6.35}$$

In these equations, $\langle . \rangle$ denotes the tidal average. It follows from Eq. (6.35) that when river flow becomes zero, so do $\hat{E}^{20}_{\text{river-river}}$ and thus no sediment will be eroded by the river flow.

7. Analytical solutions to the ordered equations

7.1 Leading order solutions of sediment concentration

From the overview of the ordered equations, it followed that the leading order sediment concentration equation is only forced by the leading order bed shear stress. In turn, the bed shear stress is a function of the leading order velocity at the bed $u^0(x,-H,t)$. Recalling that the leading order velocity only consists of an M_2 tidal signal, it thus follows that the concentration has a residual component and all tidal constituents with frequencies that are an even multiple of the M_2 tidal frequency (i.e. M_4, M_8, etc.). Hence, the solutions for u^0 and c^0, omitting contributions with frequencies higher than the M_4 frequency, can be written as

$$u^0 = \tfrac{1}{2}\hat{u}^{02}(x,z)e^{i\sigma t} + \tfrac{1}{2}\hat{u}^{02^*}(x,z)e^{-i\sigma t}, \tag{7.1}$$

$$c^0 = c^{00}(x,z) + \tfrac{1}{2}\hat{c}^{04}(x,z)e^{2i\sigma t} + \tfrac{1}{2}\hat{c}^{04^*}(x,z)e^{-2i\sigma t}, \tag{7.2}$$

Substituting the normal form of these trial solutions (the complex conjugates follow automatically) in Eq. (6.24) leads to

$$2i\sigma\hat{c}^{04}e^{2i\sigma t} - w_s\left(c^{00} + \hat{c}^{04}e^{2i\sigma t}\right)_z - K_v\left(c^{00} + \hat{c}^{04}e^{2i\sigma t}\right)_{zz} = 0. \tag{7.3}$$

Since this equation is linear we can solve for $\hat{c}^{00}$ and $\hat{c}^{04}$ separately. This is done in the next two sections.

7.1.1 Tidally averaged part of the leading order sediment concentration

The tidally averaged sediment concentration equation reads

$$K_v c^{00}_{zz} + w_s c^{00}_z = 0, \tag{7.4}$$

with boundary conditions,

$$w_s c^{00}(x,0) + K_v c^{00}_z(x,0) = 0, \tag{7.5}$$

$$-K_v c^{00}_z(x,-H) = \hat{E}^{00} a(x), \tag{7.6}$$

with $\hat{E}^{00}$ the leading order residual component of the erosion term (see Intermezzo 6.3.1). Differentiating Eq. (7.4) once with respect to z and solving the resulting differential equation leads to the following expression for c^{00},

$$c^{00} = C^{00} e^{-\frac{w_s}{K_v} z} + A, \tag{7.7}$$

where C^{00} is an amplitude and A is an integration constant. Using the boundary condition at $z = 0$, Eq. (7.5), it follows that $A = 0$. Subsequently, using the boundary condition at $z = -H$, Eq. (7.6), it follows that

$$C^{00} = \hat{E}^{00} a(x) e^{-\frac{w_s}{K_v} H}. \tag{7.8}$$

Using the expressions for A and C^{00}, the expression for c^{00} becomes

$$c^{00} = \hat{E}^{00} a(x) e^{-\frac{w_s}{K_v}(H+z)}. \tag{7.9}$$

7.1.2 M_4 part of the leading order sediment concentration

From Eq. (7.3) it follows that M_4 part of the sediment concentration equation is

$$K_v \hat{c}^{04}_{zz} + w_s \hat{c}^{04}_z - 2i\sigma \hat{c}^{04} = 0, \tag{7.10}$$

with boundary conditions,

$$w_s \hat{c}^{04}(x,0) + K_v \hat{c}^{04}_z(x,0) = 0, \tag{7.11}$$

$$-\frac{1}{2} K_v \hat{c}^{04}_z(x,-H) = \hat{E}^{04} a(x). \tag{7.12}$$

Note that now we take the M_4 harmonic component of the erosion term. The characteristic equation of Eq. (7.10) is

$$K_v r^2 + w_s r - 2i\sigma = 0,$$

and the corresponding roots are

$$r_{1,2} = \frac{-w_s \pm \lambda_{M_4}}{2K_v}, \tag{7.13}$$

with $\lambda_{M_4} = \sqrt{w_s^2 + 8i\sigma K_v}$. The general solution for $\hat{c}^{04}$ now reads

$$\hat{c}^{04} = A_1 e^{r_1 z} + A_2 e^{r_2 z}, \tag{7.14}$$

where A_1 and A_2 are unknowns that need to be determined from the boundary conditions. Substituting Eq. (7.14) into the boundary condition (7.11) for $z = 0$ leads to an expression for A_1 in terms of A_2

$$A_1 = \frac{A_2(\lambda_{M_4} - w_s)}{\lambda_{M_4} + w_s}.$$

Using the boundary condition (7.12) for $z = -H$ yields an expression for A_2,

$$A_2 = \frac{4(\lambda_{M_4} + w_s)\hat{E}^{04} a(x)}{(\lambda_{M_4} + w_s)^2 e^{-r_2 H} - (\lambda_{M_4} - w_s)^2 e^{-r_1 H}}.$$

With the expressions for A_1 and A_2 known, we can construct the solution for $\hat{c}^{04}$.

7.2 First order solutions of sediment concentration

From the overview of the ordered equations in Section , it followed that the first order sediment concentration equation is forced by the first order bed shear stress, a surface correction term because the transport across the time-dependent water surface is specified at $z = 0$ instead of the real surface $z = \zeta$, and advection of sediment (spatial settling lag effects). These forcing terms all consist of correlations between an M_2 signal and a residual plus an M_4 signal. It follows that the first order concentration has an M_2 component and all tidal constituents with frequencies that are an even multiple of the M_2 tidal frequency (i.e. M_6, M_{10}, etc.). Omitting contributions with frequencies higher than the M_4 frequency, we can write the solution for u^1 and c^1 as

$$u^1 = u^{10}(x,z) + \tfrac{1}{2}\hat{u}^{14}(x,z)e^{2i\sigma t} + \tfrac{1}{2}\hat{u}^{14^*}(x,z)e^{-2i\sigma t}, \tag{7.15}$$

$$c^1 = \tfrac{1}{2}\hat{c}^{12}(x,z)e^{i\sigma t} + \tfrac{1}{2}\hat{c}^{12^*}(x,z)e^{-i\sigma t}, \tag{7.16}$$

Because the first order system of equations presented in Section 6.3.2 is linear we can solve for each forcing mechanism separately. Hence, we can write the solution for $\hat{c}^{12}$ as

$$\hat{c}^{12} = \hat{c}^{12}_{\text{ero}} + \hat{c}^{12}_{\text{noflux}} + \hat{c}^{12}_{\text{sedadv}}.$$

The individual contributions are treated in the next sections.

7.2.1 Contribution due to bottom erosion

Substituting the trial solutions for u^0, u^1 and c^1 in Eqs. (6.28)-(6.30) and considering only the forcing term at the bottom, results in

$$K_v \left(\hat{c}^{12}_{\text{ero}}\right)_{zz} + w_s \left(\hat{c}^{12}_{\text{ero}}\right)_z - i\sigma \hat{c}^{12}_{\text{ero}} = 0, \tag{7.17}$$

with boundary conditions

$$w_s \hat{c}^{12}_{\text{ero}}(x,0) + K_v \hat{c}^{12}_{\text{ero}}(x,0) = 0, \tag{7.18}$$

$$-\tfrac{1}{2} K_v \left(\hat{c}^{12}_{\text{ero}}(x,-H)\right)_z = \hat{E}^{12} a(x). \tag{7.19}$$

Adopting the same solution method as for $\hat{c}^{04}$, we find

$$\hat{c}^{12}_{\text{ero}} = B_1 e^{r_1 z} + B_2 e^{r_2 z}, \tag{7.20}$$

with

$$r_{1,2} = \frac{-w_s \pm \lambda_{M_2}}{2K_v}, \tag{7.21}$$

$$B_1 = \frac{B_2(\lambda_{M_2} - w_s)}{\lambda_{M_2} + w_s}, \tag{7.22}$$

$$B_2 = \frac{4(\lambda_{M_2} + w_s)\hat{E}^{12} a(x)}{(\lambda_{M_2} + w_s)^2 e^{-r_2 H} - (\lambda_{M_2} - w_s)^2 e^{-r_1 H}}, \tag{7.23}$$

and $\lambda_{M_2} = \sqrt{w_s^2 + 4i\sigma K_v}$.

7.2.2 Contribution due to the no flux condition at the surface

Substituting the trial solutions for u^0, u^1 and c^1 in Eqs. (6.28)-(6.30) and considering only the forcing term due to the no flux condition at the surface, results in

$$K_v \left(\hat{c}^{12}_{\text{noflux}}\right)_{zz} + w_s \left(\hat{c}^{12}_{\text{noflux}}\right)_z - i\sigma \hat{c}^{12}_{\text{noflux}} = 0, \tag{7.24}$$

with boundary conditions

$$w_s \hat{c}^{12}_{\text{noflux}}(x,0) + K_v \hat{c}^{12}_{\text{noflux}}(x,0) = -i\sigma \hat{\zeta}^{0^*}(x)\hat{c}^{04}(x,0), \tag{7.25}$$
$$K_v \left(\hat{c}^{12}_{\text{noflux}}(x,-H)\right)_z = 0. \tag{7.26}$$

Notice that the leading order residual concentration c^{00} does not have a contribution in the forcing term at $z=0$, as it is time-independent. Furthermore, we only consider the normal form of the M_2-part of the correlation between the surface elevation and the leading order concentration amplitudes.

The derivation of the solution for $\hat{c}^{12}_{\text{noflux}}$ is equal to that of $\hat{c}^{04}$ and $\hat{c}^{12}_{\text{ero}}$ and reads

$$\hat{c}^{12}_{\text{noflux}} = C_1 e^{r_1 z} + C_2 e^{r_2 z}, \tag{7.27}$$

with r_1 and r_2 given by Eq. (7.21) and

$$C_1 = -\frac{i\sigma \hat{\zeta}^{0^*} \hat{c}^{04}_s}{(w_s + K_v r_1) - \frac{r_1}{r_2}(w_s - K_v r_2) e^{(r_2 - r_1)H}}, \tag{7.28}$$
$$C_2 = -\frac{r_1}{r_2} C_1 e^{(r_2 - r_1)H}. \tag{7.29}$$

Here, the subscript $[.]_s$ denotes the value of the physical variable evaluated at the surface.

7.2.3 Contribution due to sediment advection

Substituting the trial solutions for u^0, c^0, u^1, c^1 and additionally w^0, which is a similar expression as u^0, in Eqs. (6.28)-(6.30) and considering only the forcing term due to sediment advection, results in

$$\left(\hat{c}^{12}_{\text{sedadv}}\right)_{zz} + \frac{w_s}{K_v}\left(\hat{c}^{12}_{\text{sedadv}}\right)_z - \frac{i\sigma}{K_v}\hat{c}^{12}_{\text{sedadv}} = -\frac{\varpi}{K_v}, \tag{7.30}$$

with boundary conditions

$$w_s \hat{c}^{12}_{\text{sedadv}}(x,0) + K_v \hat{c}^{12}_{\text{sedadv}}(x,0) = 0, \tag{7.31}$$
$$K_v \left(\hat{c}^{12}_{\text{sedadv}}(x,-H)\right)_z = 0. \tag{7.32}$$

Here, the forcing term ϖ due to sediment advection is defined as

$$\varpi = \hat{u}^{02} c^{00}_x + \hat{w}^{02} c^{00}_z + \tfrac{1}{2}\left(\hat{u}^{02^*}\hat{c}^{04}_x + \hat{w}^{02^*}\hat{c}^{04}_z\right), \tag{7.33}$$

where we only consider the normal form ($\sim e^{i\sigma t}$) of the forcing term. Notice that due to the fact that the concentration amplitudes are a function of $a(x)$, the derivatives with respect to x in Eq. (7.33) additionally yield terms that are a function of a_x. The latter will be important in solving for a using the morphodynamic equilibrium condition, see Section 7.4.

Solving for $\hat{c}^{12}_{\text{sedadv}}$ from Eqs. (7.30)-(7.32) is done in the same way as for the velocity amplitude for the advection of momentum in Section 4.2.2. It involves seeking solutions for the homogeneous equation and due to the sediment advection forcing. The general expression for $\hat{c}^{12}_{\text{sedadv}}$ is

$$\hat{c}^{12}_{\text{sedadv}} = D_1 e^{r_1 z} + D_2 e^{r_2 z} + \hat{c}^{12}_{p,\text{sedadv}}, \tag{7.34}$$

with $r_{1,2}$ defined by Eq. (7.21). Furthermore, D_1 and D_2 need to be determined using the boundary conditions and $\hat{c}^{12}_{p,\text{sedadv}}$ is the particular solution due to sediment advection. The latter is found using the method of variation of parameters. Following the steps outlined in Section 4.2.2, we find

$$\hat{c}^{12}_{p,\text{sedadv}} = D_3 e^{r_1 z} - D_4 e^{r_2 z}, \tag{7.35}$$

with

$$D_3 = \int_{-H}^{z} \frac{\varpi}{\lambda_{\text{M}_2}} e^{-r_1 z'}\, dz', \tag{7.36}$$

$$D_4 = \int_{-H}^{z} \frac{\varpi}{\lambda_{\text{M}_2}} e^{-r_2 z'}\, dz'. \tag{7.37}$$

Using boundary conditions (7.31) and (7.32), we find expressions for D_1 and D_2,

$$D_1 = \frac{(w_s + K_v r_2) \int_{-H}^{0} \frac{\varpi}{\lambda_{\text{M}_2}} e^{-r_2 z'}\, dz' - (w_s + K_v r_1) \int_{-H}^{0} \frac{\varpi}{\lambda_{\text{M}_2}} e^{-r_1 z'}\, dz'}{(w_s + K_v r_1) - \frac{r_1}{r_2}(w_s + K_v r_2) e^{(r_2 - r_1)H}}, \tag{7.38}$$

$$D_2 = \frac{(w_s + K_v r_2) \int_{-H}^{0} \frac{\varpi}{\lambda_{\text{M}_2}} e^{-r_2 z'}\, dz' - (w_s + K_v r_1) \int_{-H}^{0} \frac{\varpi}{\lambda_{\text{M}_2}} e^{-r_1 z'}\, dz'}{(w_s + K_v r_2) - \frac{r_2}{r_1}(w_s + K_v r_1) e^{(r_1 - r_2)H}}. \tag{7.39}$$

The full solution for $\hat{c}^{12}_{\text{sedadv}}$ now reads

$$\hat{c}^{12}_{\text{sedadv}} = (D_1 + D_3) e^{r_1 z} + (D_2 - D_4) e^{r_2 z}, \tag{7.40}$$

with the variables D_1-D_4 defined above.

7.3 Second order solution of sediment concentration due to erosion by river flow

The solution for the sediment concentration $c^{20}_{\text{river-river}}$ as a result of erosion by river flow is equal to that of the tidally averaged part of the leading order sediment concentration c^{00} and reads

$$c^{20}_{\text{river-river}} = \hat{E}^{20}_{\text{river-river}} a(x) e^{-\frac{w_s}{K_v}(H+z)}, \tag{7.41}$$

where $\hat{E}^{20}_{\text{river-river}}$ is given by Eq. (6.35).

7.4 Morphodynamic equilibrium condition; transport components

In the previous sections we derived the amplitudes of the sediment concentration due to various physical processes. These amplitudes are functions of the still unknown sediment availability $a(x)$. To calculate the actual sediment concentrations we use the morphodynamic equilibrium condition (6.6) to solve for $a(x)$, which resulted in the following differential equation for a,

$$Fa_x + Ta = 0.$$

Since we know the concentration amplitudes, we can obtain exact analytical expressions for F and T that each contain several contributions that represent either an advective or a diffusive transport process. This allows us to assess each process separately, which highly simplifies the analysis of sediment dynamics in estuaries. Below these different transport processes are given.

For readability we repeat the expressions for F and T below,

$$F = \left\langle \int_{-H}^{0} \frac{uc^{ax}}{a_x} - K_h \frac{c^a}{a} dz \right\rangle,$$
$$T = \left\langle \int_{-H}^{0} \left(\frac{uc^a}{a} - K_h \left(\frac{c^a}{a} \right)_x \right) dz + \zeta^0 u^0(x,0) c^0(x,0) \right\rangle.$$

The extra term that arises in the expression for T is called the *Stokes drift* and is a result of the fact that during flood more water is transported in a landward direction than is transported back to the seaward direction during ebb. The diffusive transport contributions of F consist of the diffusion by the tide and river, i.e. $K_h c^{00}$ and $K_h c^{20}_{\text{river}}$, respectively. In addition, through the forcing ϖ (Eq. (7.33)), the sediment advection contribution has a diffusive part, $\hat{u}^{02}\hat{c}^{12,ax}_{\text{sedadv}}$. Here, the superscript $[.]^{ax}$ denotes the part of the concentration that is a function of a_x. The contributions are thus,

$$F_{\text{sedadv}} = \int_{-H}^{0} \frac{1}{4a_x} \left[\hat{u}^{02} \hat{c}_{\text{sedadv}}^{12^*,ax} + \hat{u}^{02^*} \hat{c}_{\text{sedadv}}^{12,ax} \right] dz, \tag{7.42}$$

$$F_{\text{diffusion-tide}} = -\int_{-H}^{0} K_h \frac{c^{00}}{a} dz, \tag{7.43}$$

$$F_{\text{diffusion-river}} = -\int_{-H}^{0} K_h \frac{c^{20}_{\text{river}}}{a} dz. \tag{7.44}$$

The advective transport contributions of T stem mainly from interactions between the first order velocity and leading order concentration, $u^1 c^0$, and/or between the leading order velocity and first order concentration, $u^0 c^1$. Additionally, there are diffusive components by the tide and the river, $K_h c_x^{00}$ and $K_h c^{20}_{x,\text{river}}$, a component by the erosion and consequent transport by the river flow, $u^{10}_{\text{river}} c^{20}_{\text{river}}$ and a component due to the Stokes drift, $\zeta^0 u^0(x,0) c^0(x,0)$. In Table 7.1 the general expressions for each contribution of the advective transport is given.

Table 7.1: Transport mechanisms

Transport mechanism	**Harmonic**	**Forcing**	**Expression**
Velocity asymmetry	residual	river baroclinic advection no stress Stokes	$\int_{-H}^{0} u^{10} \frac{c^{00}}{a} dz$
	M_4	ext. M_4 tide advection Stokes no stress	$\int_{-H}^{0} \frac{1}{4a} \left[\hat{u}^{14} \hat{c}^{04^*} + \hat{u}^{14^*} \hat{c}^{04} \right] dz$
Erosion asymmetry	M_2	river baroclinic advection no stress Stokes ext. M4 tide no flux sed. adv.	$\int_{-H}^{0} \frac{1}{4a} \left[\hat{u}^{02} \hat{c}^{12a^*} + \hat{u}^{02^*} \hat{c}^{12a} \right] dz$
Diffusion	residual	M_2 tide	$-\int_{-H}^{0} K_h \left(\frac{c^{00}}{a} \right)_x dz$
		river	$-\int_{-H}^{0} K_h \left(\frac{c^{20}_{\text{river}}}{a} \right)_x dz$
River-river interaction	residual	river	$\int_{-H}^{0} u^{10}_{\text{river}} \frac{c^{20}_{\text{river}}}{a} dz$
Stokes drift	residual	M_2 tide	$\left[\hat{\zeta}^0 \hat{u}^{02^*} \frac{c^{00}}{a} + \hat{\zeta}^{0^*} \hat{u}^{02} \frac{c^{00}}{a} \right]_{z=0}$
	M_4	M_2 tide	$\left[\hat{\zeta}^0 \hat{u}^{02} \frac{\hat{c}^{04^*}}{a} + \hat{\zeta}^{0^*} \hat{u}^{02^*} \frac{\hat{c}^{04}}{a} \right]_{z=0}$

Finally, when we substitute the known expressions for F and T into the availability expression, Eq. (6.10), we obtain the availability a and thus the final expressions for the concentration

c and its underlying components.

8. Numerical implementation

This chapter describes the numerical implementation of the leading and first order sediment dynamics of the semi-analytical iFlow model package discussed in Chapters 6 and 7. In the code itself, documentation is present that describes the module and each function used to calculate sediment transport and concentration. Here, an overview of the structure of sediment dynamics module is given.

8.1 General structure of the sediment dynamics module

The sediment dynamics module can be found in the package folder of iFlow, e.g. for version 2.4 the pathname is

```
../packages/semi_analytical2DV/sediment/
```

The module is designed as a class in which several functions are defined; the compulsory `__init__()` and `run()` functions (see iFlow modelling framework manual) together with ones that actually calculate the sediment transport functions and concentration components. The function structure and explanation for each module are given in Table 8.1.

The module first loads parameter values and calculated physical variables from the DataContainer. Subsequently, it systematically runs through the steps to calculate the sediment concentration. First, the sediment concentration amplitudes are calculated. From those, the transport function T and diffusion function F and sediment availability a are calculated. Finally, the sediment concentration components are computed. Note that the module only calculates those sediment concentration components that are given by the user on input.

Table 8.1: Function structure of the class *SedDynamic*.

SedDynamic
- **run**: Builds the output dictionary with the concentration amplitudes $\hat{c}$, the transport and diffusion functions T and F, the availability a and the concentrations c. - **erosion_lead**: Calculates the erosion induced leading order sediment concentration amplitudes c^{00} and $\hat{c}^{04}$ and their derivatives w.r.t. x and z; Eqs. (7.9) and (7.14). - **erosion**: Calculates the erosion induced first order sediment concentration amplitude c^{12}_{ero}; Eq. (7.20). - **erosion_second**: Calculates the erosion induced second order sediment concentration amplitude c^{20}_{river} by the river flow; Eq. (7.41). - **noflux**: Calculates the first order sediment concentration amplitude c^{12}_{noflux} due to the no flux surface boundary condition; Eq. (7.27). - **sedadv**: Calculates the first order sediment concentration amplitude c^{12}_{noflux} due to sediment advection; Eq. (7.40). - **availability**: Calculates the availability of sediment; Eqs. (6.10)-(6.12). - **dictExpand**: Adds a maximum of two layers to a dictionary. Mainly used to build the transport dictionary.

Bibliography

Brouwer, R. (2017). *iFlow modelling framework. Semi-analytical 2DV perturbation model package for iFlow.*

Cheng, P., Valle-Levinson, A., and de Swart, H. E. (2010). Residual Currents Induced by Ammetric Tidal Mixing in Weakly Stratified Narrow Estuaries. *J. Phys. Oceanogr.*, 40:2135–2147.

Chernetsky, A., Schuttelaars, H., and Talke, S. (2010). The effect of tidal asymmetry and temporal settling lag on sediment trapping in tidal estuaries. *Ocean Dynamics*, 60(5):1219–1241.

Chernetsky, A. S. (2012). *Trapping of sediment in tidal estuaries.* PhD thesis, Delft University of Technology.

de Swart, H. and Zimmerman, J. (2009). Morphodynamics of tidal inlet systems. *Ann. Rev. Fluid Mech.*, 41:203–229.

Dijkstra, Y. (2017a). *iFlow modelling framework. Numerical 2DV perturbation model package for iFlow.*

Dijkstra, Y. (2017b). *iFlow modelling framework. User manual & technical description.*

Dijkstra, Y., Brouwer, R., Schuttelaars, H., and Schramkowski, G. (Manuscript submitted to Geoscientific Model Development). The iflow modelling framework v2.4. a modular idealised processes model for flow and transport in estuaries.

Friedrichs, C. T., Armbrust, B. D., and De Swart, H. E. (1998). Hydrodynamics and equilibrium sediment dynamics of shallow, funnel–shaped tidal estuaries. In Dronkers, J. and Scheffers, M., editors, *Physics of Estuaries and Coastal Seas*, pages 337–343, Rotterdam. PECS 1996, Balkema.

Friedrichs, C. T. and Hamrick, J. M. (1996). Effects of channel geometry on cross section variation in along channel velocity in partially stratified estuaries. In Aubrey, D. G. and Friedrichs, C. T., editors, *Buoyancy effects on coastal and estuarine dynamics*, pages 283–300, Washington, D.C. AGU.

Groen, P. (1967). On the residual transport of suspended matter by an alternating tidal current. *Neth. J. Sea Res.*, 3:564–575.

Ianniello, J. P. (1977). Tidally induced residual currents in estuaries of constant breadth and depth. *J. Mar. Res.*, 35:755–786.

Ianniello, J. P. (1979). Tidally induced residual currents in estuaries of variable breadth and depth. *J. Phys. Oc.*, 9:962–974.

Murray, A. B. (2003). Contrasting the goals, strategies, and predictions associated with simplified numerical models and detailed simulations. In Wilcock, P. R. and Iverson, R. M., editors, *Prediction in Geomorphology*, volume 135 of *Geophysical Monograph Series*, pages 151–165, Washington D.C. AGU.

Postma, H. (1954). Hydrography of the Dutch Wadden Sea. *Arch. Neerl. Zool.*, 10:405–511.

Schramkowski, G., Schuttelaars, H. M., and de Swart, H. E. (2002a). The effect of geometry and bottom friction on local bed forms in a tidal embayment. *Continental Shelf Research*, 22:1821–1833.

Schramkowski, G. P., Schuttelaars, H. M., and de Swart, H. E. (2002b). The effect of geometry and bottom friction on local bed forms in a tidal embayment. *Cont. Shelf Res.*, 22:1821–1833.

Stacey, M. T., Brennan, M. L., Burau, J. R., and Monismith, S. G. (2010). The tidally averaged momentum balance in a partially and periodically stratified estuary. *J. Phys. Oceanogr.*, 40:2418–2434.

Stacey, M. T., Burau, J. R., and Monismith, S. G. (2001). Creation of residual flows in a partially stratified estuary. *J. Geophys. Res.*, 106:17013–17037.

Talke, S. A., de Swart, H. E., and de Jonge, V. N. (2009). An Idealized Model and Systematic Process Study of Oxygen Depletion in Highly Turbid Estuaries. *Estuaries and Coasts*, 32(4):602–620.

Warner, J. C., Geyer, W. R., and Lerczak, J. A. (2005). Numerical modeling of an estuary: A comprehensive skill assessment. *J. Geophys. Res.*, 110:C05001.

Zimmerman, J. T. F. (1982). On the Lorentz linearization of a quadratically damped forced oscillator. *Phys. Lett.*, 89A:123–124.