A class of new explicit Runge-Kutta schemes*

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Abstract An improvement of the traditional Runge-Kutta schemes, a species of dissipative schemes, produces a class of new square conservative schemes with the same scheme formats and precisions, yielding results as good as those from the symplectic schemes in the numerical tests, and vests this old antiquated algorithm with new life.

Keywords: Runge-Kutta scheme, square conservation, precision.

The Runge-Kutta method, a classic numerical method, has ever played an important role in simulations and computations in many fields of science. The progress in science and technology, however, makes it hard to meet the needs of scientific and engineering computations. For example, in solutions to square conservation systems or Hamilton systems by this method, great distortions turn up, especially in those of long-time integrations by this method, which is verified in an instance in ref. [1]. Should the method be eliminated completely? No. A careful analysis on it shows that the Runge-Kutta method owns three distinct properties: first, there is no limit in heightening the precision of the method; next, the absolute stability interval of it may be lengthened when the precision rises; last, it is an explicit and single step method. These properties are worth keeping and using. They have been successfully applied to construct a species of high-order consistent dissipation operators[2], which greatly improve the stabilities and precisions of the explicit square conservative schemes[2—4] and produce obvious benefits in practical applications[5, 6]. In this paper, the thought of square conservative schemes is poured into the Runge-Kutta method, and a class of new explicit schemes, keeping the quadratic conservation properties and being consistent with this method in precision, stabiblity and scheme format, are constructed. Good results are obtained from the new schemes in numerical tests. The new study make the old method reborn.

1 The construction of the schemes

Consider the nonlinear evolution equation:

$$\frac{\partial F}{\partial t} = LF. \tag{1}$$

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If the operator L is antisymmetrical:

$$(LF,\ F)=0, \tag{2}$$

then eq. (1) is of square conservation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|F\|^2=0. \tag{3}$$

The k-th order explicit Runge-Kutta schemes to solve eq. (1) can be expressed as

$$F^{n+1}=F^n+\Delta t\varphi(F^n,\ \Delta t), \tag{4}$$

where

$$\varphi(F^n,\ \Delta t)=\sum_{i=1}^{k}C_iR_i \tag{5}$$

and

$$\begin{cases} R_1=LF^n, \\ R_i=L\left(F^n+\Delta t\sum_{j=1}^{i-1}b_{ij}R_j\right) \quad (i=2,\ 3,\cdots,\ K). \end{cases} \tag{6}$$

These are dissipative schemes[1] and they are unable to keep the conservation properties. Therefore, solutions to square conservation systems from them may be greatly distorted. How can these schemes be improved? Here, a method to make them quadratically conservative is introduced. For this aim, eq. (1) is corrected into:

$$F^{n+1}=F^n+\tau_n\varphi(F^n,\ \Delta t), \tag{7}$$

i.e. the integration interval of the original schemes Δt is changed into an adjustable interval τ_n, while that in φ is invariable. τ_n can be determined by using the square conservation properties. Scheme (7) is hoped to satisfy under the antisymmetrical condition (2):

$$\|F^{n+1}\|^2=\|F^n\|^2 \tag{8}$$

and it may be expressed as (mark $\varphi^n=\varphi(F^n,\ \Delta t)$)

$$\tau_n=-\frac{2(\varphi^n,\ F^n)}{\|\varphi^n\|^2}. \tag{9}$$

However, if τ_n is just determined by eq. (9), scheme (7) is compatible with eq. (1) only when condition (2) is satisfied. The compatibility will be proved in the coming discussion. When eq. (2) is not true, eq. (7) becomes incompatible with eq. (1). This is because $\tau_n\to-\dfrac{2(LF^n,\ F^n)}{\|F^n\|^2}\neq 0$ when $\Delta t\to 0$. Therefore, eq. (9) is not a suitable method of determining τ_n and should be replaced by a better way. According to eq. (5),

$$(\varphi^n,\ F^n)=\sum_{i=1}^{k}C_i(R_i,\ F^n). \tag{10}$$

If we mark

$$F_i^{n+1}=F^n+\Delta t\sum_{j=1}^{i-1}b_{i,j}R_j \quad (j=2,\ 3,\cdots,\ k), \tag{11}$$

then eq. (6) becomes

$$R_i=LF_i^{n+1} \quad (R_1=LF^n,\ i=2,\ 3,\cdots,\ k). \tag{12}$$

According to eq. (2),

$$(LF_i^{n+1},\ F_i^{n+1})=0 \tag{13}$$

and

$$\left(R_i,\ F^n+\Delta t\sum_{j=1}^{i-1}b_{i,j}R_j\right)=0, \tag{14}$$

which deduces that

$$(R_i,\ F^n)=-\Delta t\sum_{j=1}^{i-1}b_{i,j}(R_i,\ R_j). \tag{15}$$

By substituting eq. (15) into eq. (10), we have

$$(\varphi^n,\ F^n)=-\Delta t\sum_{i=1}^{k}C_i\sum_{j=1}^{i-1}b_{i,j}(R_i,\ R_j). \tag{16}$$

In this way, another form to determine τ_n is established by substituting eq. (16) into eq. (9)

$$\tau_n=\beta_n\Delta t, \tag{17}$$

where

$$\beta_n=\frac{2}{||\varphi^n||^2}\sum_{i=1}^{k}C_i\sum_{j=1}^{i-1}b_{i,j}(R_i,\ R_j). \tag{18}$$

Finally, scheme (7) determined by eqs. (17) and (18) is just the new explicit Runge-Kutta method we want to construct.

2 Analysis on precisions and conservations

Theorem 1. *Under the antisymmetrical condition* (2), *scheme* (7) *determined by eqs.* (17) *and* (18) *is of square conservation and owns k-th order precision.*

Proof. Because eqs. (17) and (18) are equivalent to eq. (9) under the antisymmetrical condition, it is obvious that scheme (7) is of square conservation. For this reason, only the precision of the scheme is discussed here. Suppose F^n is the accurate solution $F^n=(F)^n$, then

$$(F)^{n+1}=(F)^n+\Delta t(\varphi)^n+O(\Delta t^{k+1})=F^n+\Delta t\varphi^n+O(\Delta t^{k+1}). \tag{19}$$

The norm operation on eq. (19) and the square conservation property (3) produce

$$\Delta t||\varphi^n||^2+2(\varphi^n,\ F^n)=O(\Delta t^k), \tag{20}$$

and the property that eqs. (17)—(18) are equivalent to eq. (9) under the antisymmetrical condition deduces the following equality

$$\tau_n \|\varphi^n\|^2 + 2(\varphi^n, F^n) = 0. \tag{9$'$}$$

Eq. (9)′ minus eq. (20) is

$$(\tau_n - \Delta t)\|\varphi^n\|^2 = O(\Delta t^k), \tag{21}$$

i.e.

$$\tau_n = \Delta t + O(\Delta t^k) \tag{21$'$}$$

or

$$\Delta t = \tau_n + O(\Delta t^k), \ \beta_n = \frac{\tau_n}{\Delta t} = 1 + O(\Delta t^{k-1}). \tag{22}$$

Because $LF = G(F, F_x, F_y, F_z)$ in general, where G is a sufficiently smooth function, it is testisfied that

$$\varphi(F^n, \Delta t) = \varphi(F^n, \tau_n + O(\Delta t^k)) = \varphi(F^n, \tau_n) + O(\Delta t^k). \tag{23}$$

In this way,

$$\frac{(F)^{n+1} - (F)^n}{\tau_n} - (\varphi)^n = \frac{(F)^{n+1} - (F)^n}{\tau_n} - \varphi((F)^n, \tau_n) + O(\Delta t^k) = O(\tau_n^k) + O(\Delta t^k) = O(\Delta t^k). \tag{24}$$

Therefore, the scheme is of k-th order precision.

Theorem 2. *If the antisymmetrical condition* (2) *is not true, scheme* (9) *determined by eqs.* (17) and (18) *is still of k-th order precision.*

Due to the complication in determining the coefficients of the Runge-Kutta method, it is difficult to give a general proof of the theorem. Only some verifications in simple situations such as $k=2$ and $k=3$ are given here, for which it suffices only to testify the estimation expression $\beta_n = \frac{\tau_n}{\Delta t} = 1 + O(\Delta t^{k-1})$.

Now, the verification in the case of $k=2$ is shown first. In this situation, the original Rung-Kutta method is

$$\varphi(F, \Delta t) = \varphi_2 = c_1 R_1 + c_2 R_2, \tag{25}$$

$$\begin{cases} R_1 = LF, \\ R_2 = L(F + \Delta t b_{21} R_1), \end{cases} \tag{26}$$

where c_1, c_2 and b_{21} satisfy

$$\begin{cases} c_1 + c_2 = 1, \\ c_2 b_{21} = \dfrac{1}{2}. \end{cases} \tag{27}$$

For the equations of atmosphere and ocean, LF is generally a function depending on F and its first-, second- and higher-order spatial partial differential coefficients are:

$$LF=G(F,\ F_x,\ F_y,\ F_z,\ F_{x^2},\cdots). \tag{28}$$

In order that the discussion is simple and clear, it is representively set that

$$LF=G(F,\ F_x), \tag{28$'$}$$

where $G=G(\zeta,\ \eta)$ is sufficiently smooth with $\zeta,\ \eta$. By using the Taylor expansion, we have

$$R_2=L(F+\Delta t b_{21}R_1)=G\left(F+\Delta t b_{21}R_1,\ F_x+\Delta t b_{21}\frac{\partial R_1}{\partial x}\right)=G(F,\ F_x)$$

$$+\Delta t b_{21}\left[G_\zeta R_1+G_\eta\frac{\partial R_1}{\partial x}\right]+O(\Delta t^2),$$

which can be simplified into

$$R_2=\frac{\partial F}{\partial t}+\Delta t b_{21}\frac{\partial^2 F}{\partial t^2}+O(\Delta t^2)$$

due to the following two equalities

$$\frac{\partial F}{\partial t}=LF=G(F,\ F_x),$$

$$\frac{\partial^2 F}{\partial t^2}=G_\zeta\frac{\partial F}{\partial t}+G_\eta\frac{\partial F_x}{\partial t}=G_\zeta R_1+G_\eta\frac{\partial R_1}{\partial x}. \tag{29}$$

From eqs. (25), (26) and (27), it is easily deduced that

$$\varphi=\frac{\partial F}{\partial t}+\frac{1}{2}\Delta t\frac{\partial^2 F}{\partial t^2}+O(\Delta t^2)$$

and

$$\|\varphi\|^2=\left\|\frac{\partial F}{\partial t}\right\|^2+\Delta t\left(\frac{\partial^2 F}{\partial t^2},\ \frac{\partial F}{\partial t}\right)+O(\Delta t^2). \tag{30}$$

In eq. (18), mark

$$\beta'=2\sum_{i=1}^{k}c_i\sum_{j=1}^{i-1}b_{i,j}(R_i,\ R_j), \tag{31}$$

then

$$\beta'=2c_2b_{21}(R_2,\ R_1)=(R_2,\ R_1)=\left\|\frac{\partial F}{\partial t}\right\|^2+\Delta t b_{21}\left(\frac{\partial^2 F}{\partial t^2},\ \frac{\partial F}{\partial t}\right)+O(\Delta t^2)$$

and

$$\beta_n=\frac{\beta'}{\|\varphi^n\|^2}=1+(b_{21}-1)\Delta t\frac{\left(\frac{\partial^2 F}{\partial t^2},\ \frac{\partial F}{\partial t}\right)}{\left\|\frac{\partial F}{\partial t}\right\|^2}+O(\Delta t^2),$$

when $b_{21} \neq 1$, $\beta_n = 1 + O(\Delta t)$, and scheme (7) is of second-order precision. If $b_{21} = 1$, the scheme is still of second-order precision, although $\beta_n = 1 + O(\Delta t^2)$, because a second-order Runge-Kutta scheme is only of second-order precision. While $k \geqslant 3$, there hardly exists the possibility that $\beta_n = 1 + O(\Delta t^k)$.

Next, the situation that $k=3$ is considered

$$\varphi(F, \ \Delta t) = \varphi_3 = c_1 R_1 + c_2 R_2 + c_3 R_3,$$

where R_1 and R_2 are determined by eq. (26), R_3 satisfies

$$R_3 = L(F + \Delta t (b_{31} R_1 + b_{32} R_2)) \tag{32}$$

and c_1, c_2, c_3, b_{21}, b_{31}, b_{32} obey

$$\begin{cases} c_1 + c_2 + c_3 = 1, \ c_2 b_{21} + c_3 (b_{31} + b_{32}) = \dfrac{1}{2}, \\ c_2 b_{21}^2 + c_3 (b_{31} + b_{32})^2 = \dfrac{1}{3}, \ c_3 b_{21} b_{32} = \dfrac{1}{6}. \end{cases} \tag{33}$$

On the one hand, under the supposition of eq. (28)′, it is deduced that

$$\frac{\partial^3 F}{\partial t^3} = \left[G_{\xi^2} R_1^2 + 2 G_{\xi\eta} R_1 \frac{\partial R_1}{\partial x} + G_{\eta^2} \left(\frac{\partial R_1}{\partial x} \right)^2 \right] + \left[(G_\xi)^2 R_1 + 2 G_\xi G_\eta \frac{\partial R_1}{\partial x} + (G_\eta)^2 \frac{\partial^2 R_1}{\partial x^2} \right]$$
$$+ \left[G_\eta \frac{\partial G_\xi}{\partial x} R_1 + G_\eta \frac{\partial G_\eta}{\partial x} \frac{\partial R_1}{\partial x} \right] = P_1 + P_2 + P_3,$$

where

$$\begin{cases} P_1 = \left[G_{\xi^2} R_1^2 + 2 G_{\xi\eta} R_1 \dfrac{\partial R_1}{\partial x} + G_{\eta^2} \left(\dfrac{\partial R_1}{\partial x} \right)^2 \right], \\ P_2 = \left[(G_\xi)^2 R_1 + 2 G_\xi G_\eta \dfrac{\partial R_1}{\partial x} + (G_\eta)^2 \dfrac{\partial^2 R_1}{\partial x^2} \right], \\ P_3 = \left[G_\eta \dfrac{\partial G_\xi}{\partial x} R_1 + G_\eta \dfrac{\partial G_\eta}{\partial x} \dfrac{\partial R_1}{\partial x} \right]. \end{cases} \tag{34}$$

Consequently,

$$R_2 = G\left(F + \Delta t b_{21} R_1, \ F_x + \Delta t b_{21} \frac{\partial R_1}{\partial x} \right) = R_1 + \Delta t b_{21} \left[G_\xi R_1 + G_\eta \frac{\partial R_1}{\partial x} \right]$$
$$+ \frac{1}{2} \Delta t^2 b_{21}^2 \left[G_{\xi^2} R_1^2 + 2 G_{\xi\eta} R_1 \frac{\partial R_1}{\partial x} + G_{\eta^2} \left(\frac{\partial R_1}{\partial x} \right)^2 \right] + O(\Delta t^3)$$
$$= \frac{\partial F}{\partial t} + \Delta t b_{21} \frac{\partial^2 F}{\partial t^2} + \frac{1}{2} \Delta t^2 b_{21}^2 P_1 + O(\Delta t^3),$$

$$R_3=G\left(F+\Delta t b_{31}R_1+\Delta t b_{32}R_2,\ F_x+\Delta t b_{31}\frac{\partial R_1}{\partial x}+\Delta t b_{32}\frac{\partial R_2}{\partial x}\right)$$

$$=G(F,\ F_x)+\Delta t\left[(b_{31}R_1+b_{32}R_2)G_\xi+\left(b_{31}\frac{\partial R_1}{\partial x}+b_{32}\frac{\partial R_2}{\partial x}\right)G_\eta\right]$$

$$+\frac{\Delta t^2}{2}\left[G_{\xi^2}(b_{31}R_1+b_{32}R_2)^2+G_\eta\left(b_{31}\frac{\partial R_1}{\partial x}+b_{32}\frac{\partial R_2}{\partial x}\right)^2\right.$$

$$\left.+2G_{\xi\eta}(b_{31}R_1+b_{32}R_2)\left(b_{31}\frac{\partial R_1}{\partial x}+b_{32}\frac{\partial R_2}{\partial x}\right)\right]+O(\Delta t^3)$$

$$=R_1+\Delta t(b_{31}+b_{32})\left(G_\xi R_1+G_\eta\frac{\partial R_1}{\partial x}\right)$$

$$+\Delta t^2 b_{32}b_{21}\left\{\left[G_\xi^2R_1+2G_\xi G_\eta\frac{\partial R_1}{\partial x}+G_\eta^2\frac{\partial^2 R_1}{\partial x^2}\right]+\left[G_\eta\frac{\partial G_\xi}{\partial x}R_1+G_\eta\frac{\partial G_\eta}{\partial x}\frac{\partial R_1}{\partial x}\right]\right\}$$

$$+\frac{\Delta t^2}{2}(b_{31}+b_{32})^2\left[G_{\xi^2}R_1^2+2G_{\xi\eta}R_1\frac{\partial R_1}{\partial x}+G_{\eta^2}\left(\frac{\partial R_1}{\partial x}\right)^2\right]+O(\Delta t^3)$$

$$=\frac{\partial F}{\partial t}+\Delta t(b_{31}+b_{32})\frac{\partial^2 F}{\partial t^2}+\Delta t^2\left[b_{32}b_{31}(P_2+P_3)+\frac{1}{2}(b_{31}+b_{32})^2P_1\right]+O(\Delta t^3)$$

and

$$\varphi=\frac{\partial F}{\partial t}+\frac{1}{2}\Delta t\frac{\partial^2 F}{\partial t^2}+\frac{1}{6}\Delta t^2\frac{\partial^3 F}{\partial t^3}+O(\Delta t^3)$$

from eq. (33), which shows that

$$\|\varphi\|^2=\left\|\frac{\partial F}{\partial t}\right\|^2+\Delta t\left(\frac{\partial^2 F}{\partial t^2},\ \frac{\partial F}{\partial t}\right)+\Delta t^2\left[\frac{1}{4}\left\|\frac{\partial^2 F}{\partial t^2}\right\|^2+\frac{1}{3}\left(\frac{\partial^3 F}{\partial t^3},\ \frac{\partial F}{\partial t}\right)\right]+O(\Delta t^3).$$

On the other hand,

$$(R_2,\ R_1)=\left\|\frac{\partial F}{\partial t}\right\|^2+\Delta t b_{21}\left(\frac{\partial^2 F}{\partial t^2},\ \frac{\partial F}{\partial t}\right)+\frac{\Delta t^2}{2}b_{21}^2\left(P1,\ \frac{\partial F}{\partial t}\right)+O(\Delta t^3),$$

$$(R_3,\ R_1)=\left\|\frac{\partial F}{\partial t}\right\|^2+\Delta t(b_{31}+b_{32})\left(\frac{\partial^2 F}{\partial t^2},\ \frac{\partial F}{\partial t}\right)+\Delta t^2 b_{32}b_{21}\left[\left(P_2,\ \frac{\partial F}{\partial t}\right)+\left(P_3,\ \frac{\partial F}{\partial t}\right)\right]$$

$$+\frac{\Delta t^2}{2}(b_{31}+b_{32})^2\left(P_1,\ \frac{\partial F}{\partial t}\right)+O(\Delta t^3)$$

and

$$(R_3,\ R_2)=\left\|\frac{\partial F}{\partial t}\right\|+\Delta t(b_{21}+b_{31}+b_{32})\left(\frac{\partial^2 F}{\partial t^2},\ \frac{\partial F}{\partial t}\right)$$

$$+\Delta t^2 b_{32} b_{21}\left[\left(P_2,\ \frac{\partial F}{\partial t}\right)+\left(P_3,\ \frac{\partial F}{\partial t}\right)\right]+\frac{\Delta t^2}{2}\left[(b_{31}+b_{32})^2+b_{21}^2\right]\left(P_1,\ \frac{\partial F}{\partial t}\right)$$

$$+\Delta t^2 b_{21}(b_{31}+b_{32})\left\|\frac{\partial^2 F}{\partial t^2}\right\|^2+O(\Delta t^3).$$

By using the above three expressions, we have

$$\beta'=2[c_2 b_{21}(R_2,\ R_1)+c_3 b_{31}(R_3,\ R_1)+c_3 b_{32}(R_3,\ R_2)]=2[c_2 b_{21}+c_3(b_{31}+b_{32})]\left\|\frac{\partial F}{\partial t}\right\|^2$$

$$+2[c_2 b_{21}^2+c_3(b_{31}+b_{32})^2+c_3 b_{32} b_{21}]\Delta t\left(\frac{\partial^2 F}{\partial t^2},\ \frac{\partial F}{\partial t}\right)$$

$$+[c_2 b_{21}^3+c_3(b_{31}+b_{32})^3+c_3 b_{32} b_{21}^2]\Delta t^2\left(P_1,\ \frac{\partial F}{\partial t}\right)$$

$$+2c_3 b_{32} b_{21}(b_{31}+b_{32})\Delta t^2\left[\left(P_2,\ \frac{\partial F}{\partial t}\right)+\left(P_3,\ \frac{\partial F}{\partial t}\right)\right]$$

$$+2c_3 b_{32} b_{21}(b_{31}+b_{32})\Delta t^2\left\|\frac{\partial^2 F}{\partial t^2}\right\|^2+O(\Delta t^3)$$

and finally, the estimation expression

$$\beta'=\left\|\frac{\partial F}{\partial t}\right\|^2+\Delta t\left(\frac{\partial^2 F}{\partial t^2},\ \frac{\partial F}{\partial t}\right)+[c_2 b_{21}^3+c_3(b_{31}+b_{32})^3+c_3 b_{32} b_{21}^2]\Delta t^2\left(P_1,\ \frac{\partial F}{\partial t}\right)$$

$$+\frac{1}{3}(b_{31}+b_{32})\Delta t^2\left[\left(\frac{\partial^3 F}{\partial t^3},\ \frac{\partial F}{\partial t}\right)-\left(P_1,\ \frac{\partial F}{\partial t}\right)\right]+\frac{1}{3}(b_{31}+b_{32})\Delta t^2\left\|\frac{\partial^2 F}{\partial t^2}\right\|^2$$

$$=\left\|\frac{\partial F}{\partial t}\right\|^2+\Delta t\left(\frac{\partial^2 F}{\partial t^2},\ \frac{\partial F}{\partial t}\right)+\frac{1}{3}(b_{31}+b_{32})\Delta t^2\left[\left(\frac{\partial^3 F}{\partial t^3},\ \frac{\partial F}{\partial t}\right)+\left\|\frac{\partial^2 F}{\partial t^2}\right\|^2\right]$$

$$+\left[c_2 b_{21}+c_3(b_{31}+b_{32})^3+c_3 b_{32} b_{21}^2-\frac{1}{3}(b_{31}+b_{32})\right]\Delta t^2\left(P_1,\ \frac{\partial F}{\partial t}\right)+O(\Delta t^3)$$

can be deduced from eq. (33), which show that no matter how to adjust the coefficients $\{c_i\}$ and $\{b_{i,j}\}$, $\beta'-\|\varphi\|^2$ is constantly a second-order small quantity $O(\Delta t^2)$, not a third or more order one. Therefore, $\beta_n=\dfrac{\beta'}{\|\varphi^n\|^2}=1+O(\Delta t^2)$, and the scheme is of third-order precision.

By using the similar method, it can also be testified that scheme (7) determined by eqs. (17) and (18) is of forth-order precision when $k=4$. In the case where k is greater than 5, it is very hard to give a verification of scheme precision, and hereby the case is not discussed. A general proof is waiting to be provided from further studies. Theorem 2 shows that scheme (7) determined by eqs. (17) and (18) is a proper scheme for solving linear or nonlinear systems, with or without square conservation properties.

3 Numerical tests

To examine the practicability of the newly established explicit Runge-Kutta method, some numerical tests are carried out.

Consider the linear ordinary differential equations:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -ay, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = bx, \end{cases} \qquad (a,\ b>0), \tag{53}$$

which can be easily proved to be a Hamilton system. The accurate solution to eq. (53) is $bx^2+ay^2=c$ (c is a constant), and keeps falling over an oval. This important geometric feature can be kept by the numerical solutions from a 100 000 000-step integrations on eq. (53) by the new Runge-Kutta schemes, but it is distorted by the original Runge-Kutta method because the solutions from the old method are gradually dissipated from an oval at the initial time into a point after the 100 000 000-step integrations (figs. 1, 2). The most interesting thing is that the time interval of the new scheme τ_n, determined by eqs. (17) and (18), keeps constant from the initial time to the 100 000 000-th step, which is 1.000 010 704 417 22 when $\Delta t=1$ (table 1). If a generalized energy is defined as $E_n=bx_n^2+ay_n^2$, all the E_n with n from 1 to 100 000 000 from the new method are the same in 12 significant digits. The errors are only of 10^{-13} order of magnitude in the 100 000 000-step integrations (table 1), which are caused only by the round-off error of the computer, but not by the scheme. However, E_n from the old method keeps attenuating and finally tends to zero

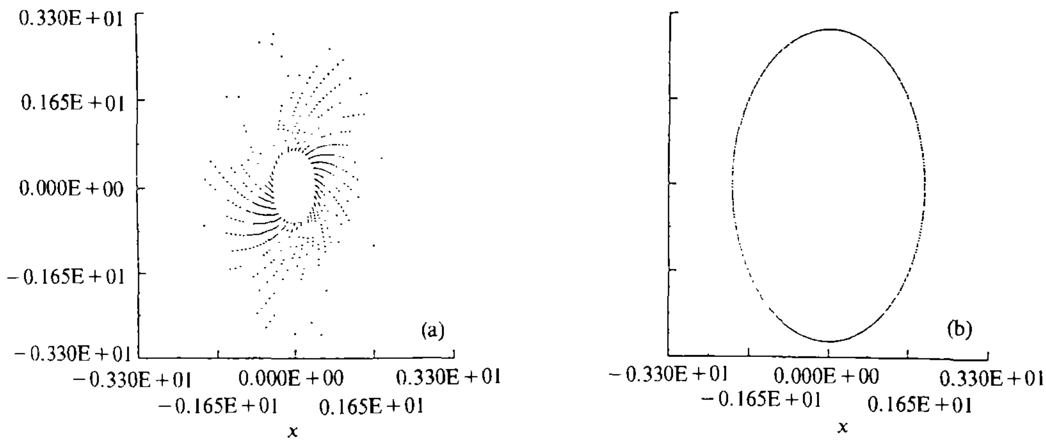

Fig. 1. Results from the 10^7-step integration (printing a result per 10^4 steps). (a) By the old Runge-Kutta scheme; (b) by the new Runge-Kutta scheme.

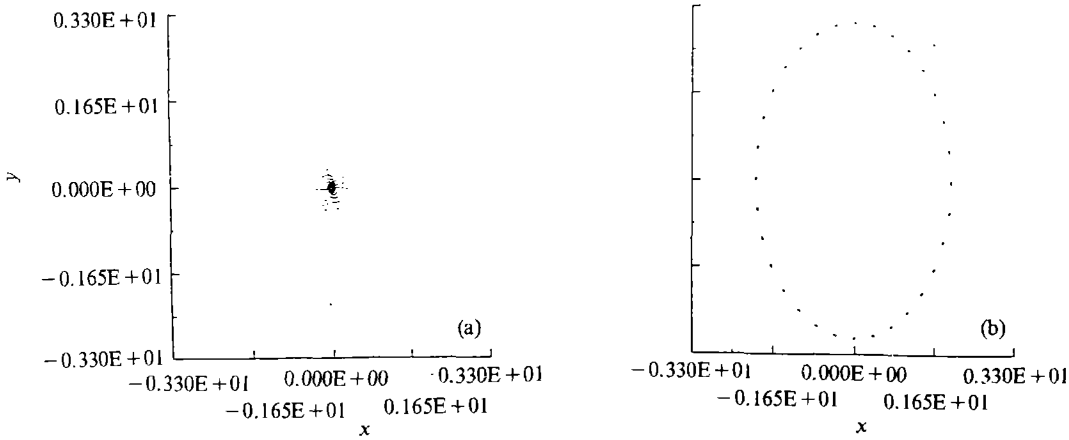

Fig. 2. Results from the 10^8-step integration (printing a result per 10^5 steps). (a) By the old Runge-Kutta scheme, (b) by the new Runge-Kutta scheme.

Table 1 Evolutions of generalized energy and the time interval of eq. (53)

Step (n)	Energy of the old Runge-Kutta scheme	Energy of the new Runge-Kutta scheme	Time interval of the new Runge-Kutta scheme
1	1.000 000 000 000 000	1.000 000 000 000 000	1.000 010 744 177 2
10^5	0.970 770 348 486 377	0.999 999 999 999 964	1.000 010 744 177 2
10^6	0.743 301 529 331 007	0.999 999 999 999 806	1.000 010 744 177 2
2×10^6	0.552 497 163 505 841	0.999 999 999 999 657	1.000 010 744 177 2
4×10^6	0.305 253 115 681 681	0.999 999 999 999 162	1.000 010 744 177 2
8×10^6	0.093 179 464 933 436	0.999 999 999 999 156	1.000 010 744 177 2
10^7	0.051 481 389 906 958	0.999 999 999 999 151	1.000 010 744 177 2
2×10^7	0.002 650 333 506 751	0.999 999 999 999 153	1.000 010 744 177 2
4×10^7	0.000 007 024 267 697	0.999 999 999 999 159	1.000 010 744 177 2
8×10^7	0.000 000 000 047 898	0.999 999 999 999 161	1.000 010 744 177 2
10^8	0.000 000 000 000 131	0.999 999 999 999 151	1.000 010 744 177 2

(table 1). Therefore, for the Runge-Kutta method, there is an essential distinction between that before the improvement and that after the improvement. Especially, results as good as the symplectic schemes are obtained from the improved method[1]. Note that the inner product used here is defined as

$$(F_1,\ F_2)=bx_1x_2+ay_1y_2, \tag{54}$$

where

$$F_i=[x_i,\ y_i]^T \quad (i=1,\ 2). \tag{54$'$}$$

In addition, another test by the new Runge-Kutta method is carried out on the spherical barotropic shallow water equations, a group of nonlinar partial differential equations. The results are the same as those from the implicit Euler central difference scheme, which is both a symplectic scheme and a square conservation scheme. The new method has much better time benefits than that of the Euler scheme. The figures of wave evolution from the new scheme is the same as those in ref. [2]. The total energy and the total mass keep conserved (table 2). Especially, when $\Delta t = 392\text{s}$, τ_n determined by eqs. (17) and (18) is very near to Δt: $|\tau_n - \Delta t| < 0.011$, and is almost invariable (table 2). Meanwhile, it is constantly true that $\tau_n > \Delta t$.

Table 2 Evolutions of the total energy, the total mass and the time interval of the Rossby-Haurwitz waves from the new Runge-Kutta scheme

Step (n)	Integration time t/s	Total energy E_n /$\text{m}^4 \cdot \text{s}^{-4}$	Total mass M_n /$\text{m}^2 \cdot \text{s}^{-2}$	Time interval τ_n/s
00 001	392.010 7	15 236 914 614 094.2	175 248 643.214 532	392.010 7
00 100	39 200.86	15 236 914 614 094.2	175 248 643.214 532	392.006 9
00 200	78 401.53	15 236 914 614 094.2	175 248 643.214 532	392.004 8
00 400	156 802.0	15 236 914 614 094.2	175 248 643.214 532	392.002 8
00 800	313 602.0	15 236 914 614 094.2	175 248 643.214 532	392.002 0
01 600	627 202.0	15 236 914 614 094.2	175 248 643.214 532	392.001 9
03 200	1 254 402.0	15 236 914 614 094.2	175 248 643.214 532	392.001 7
06 400	2 508 802.0	15 236 914 614 094.2	175 248 643.214 532	392.001 4
12 800	5 017 602.0	15 236 914 614 094.2	175 248 643.214 532	392.001 2
22 041	8 640 074.0	15 236 914 614 094.2	175 248 643.214 532	392.001 1

The above two tests show that the new Runge-Kutta method is practicable and useful, and should be greatly disseminated.

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