Dear Editor,

we respond below to the comments raised by the Reviewers to paper “SELEN4 (SELEN version 4.0): a Fortran program for solving the gravitationally and topographically self-consistent Sea Level Equation in Glacial Isostatic Adjustment modeling”, submitted to GMD (gmd-2019-183).

At the end of our responses, we have appended a marked copy of the manuscript showing the changes made.

Best regards

Giorgio Spada and Daniele Melini

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Reply to comments of REVIEWER 1: Volker Klemann

Dear Editor, dear Dr. Klemann,

Hereby we respond to the comments (in blue). In the annotated revised manuscript, the modifications made are marked in bold face and a label “R1-N” is found on the margin of the manuscript (where this is permitted by LaTeX), where R1 stands for Reviewer 1 and N is the point made by Reviewer 1 (labels are defined, in blue, in this letter).

We are also grateful to Dr. Klemann for having pointed out that the RSL data we have used in this work are given in uncalibrated time units, and should be calibrated before drawing any comparison with GIA models outputs. We have mentioned this issue in the text.

We hope that we have responded satisfactorily to the constructive comments received, and we are looking forward to have your feedback.

Note that we have also corrected a few typos, added some references, improved the text in a few places, and re-edited some of the figures and tables.

Kind regards

Giorgio Spada & Daniele Melini


Interactive comment on “SELEN4 (SELEN version 4.0): a Fortran program for solving the gravitationally and topographically self-consistent Sea Level Equation in Glacial Isostatic Adjustment modeling” by Giorgio Spada and Daniele Melini

Comments by Reviewer: Volker Klemann

General comments

Point R1-0
The authors present in this manuscript an update of numerical code which enables to calculate the gravitationally consistent interaction between a surface mass load and water mass load which compensates the total mass change. Previous versions of this code were presented already on a number of workshops dealing with glacial isostatic adjustment, GIA. In contrast to these versions, the authors improved the code in a number of aspects which are currently discussed in GIA, and extended the portability of the code a lot. These improvements, to my point of view justify a new publication in a method and code-oriented Journal like GMD.

As I don’t understand this review as a testing of the code, I will focus my review purely on the presented manuscript, and will not consider the supplement in this regard, especially for the derivation. In general, the manuscript is clearly written, the focus lies on the considered theory which the authors present in the theory section, and, as an application, the authors chose a published and established forcing for which they discuss the output in detail. Due to the fact, that the manuscript presents a methodologically oriented study, they do not discuss deviations from their results to those presented in the original publication of this forcing. From my point of view, this is reasonable strategy.

R. We thank Volker Klemann for his positive evaluation and for the suggestions made. We have made efforts to address all his comments; when not, a justification is given. See the details given below.

**Point R1-1**

As a validation of the method they refer to the benchmark study Martinec et al., 2018, which was recently published, and to which they contributed with a preliminary version of this code version. Of course the question may arise, why the authors did not present the results for the benchmark, and discuss the addition of the rotational effect which was not considered in that benchmark this would be a nice extension to that study. If they won’t do this, at least they should state, that the results of that version are identical with the current one, if rotation is switched off. Consulting that study, the SELEN code deviates a bit more than the order models when considering moving coastlines and floating ice. But this is only a
suggestion, to establish more a benchmark study, than the quite complicated ICE6G vm5a applied here.

R. The benchmark study in Martinec et al., 2018 was conducted between codes that do not account for rotational feedback (or in which these effects have been switched off). While we certainly could recompute benchmark results taking into account rotational effects, they cannot be directly compared with other results; actually, we could only discuss the difference between SELEN results with and without rotation. To some extent, this is actually done in the manuscript; moreover, in a recently published review paper (Spada and Melini, Water, 2019) the effect of rotational feedback on GIA fingerprints has been discussed in greater detail. In the revised manuscript, we now refer explicitly the reader to this paper. We think that a benchmark between SLE solvers that include rotational feedback would be of great interest, however at this stage we believe that recomputing the 2018 benchmark results with the inclusion of rotational feedback would not add insight to the discussion. As suggested by the Reviewer, we explicitly remark in the revised manuscript that the SELEN4 results, without rotation and in the same conditions of the 2018 benchmark, coincide with the results published in that context. See also point R2-1.

In the following, I focus on individual aspects of the presentation.

to Introduction

Point R1-2
P. 2, L 16: Elasticity is material law, but not a rheology.

The reviewer is right, we have rephrased and we do not use “elastic rheology” in the revised paper.

to Theory

Point R1-3
To present a reduced version of a derivation is every time dangerous, especially if this is not put into an appendix but a supplement which is not
part of the manuscript. So, a number of questions arose, which partly come up in the following comments.

We are aware that balancing detail and conciseness in a theoretical section is not an easy task. We thank the reviewer for his suggestions on this point. We agree that the supplement is not strictly part of the manuscript, but it is nevertheless available to the reader after publication of the manuscript.

**Point R1-4**  
P. 4, L 7: You refer to SSM19, here you should at least specify on which principles this expression (Eq. 2) is based.

We have introduced an explicit expression for the load (L) in terms of mass per unit area, which should illustrate better the physical meaning of L. Eq. (3) (formerly Eq. 2) is now better framed since we have made explicit the ocean function, following the suggestion in R1-5.

**Point R1-5**  
P. 4, L 10: The meaning of OF should be defined explicitly.

This has now been done, since it effectively helps a lot the understanding of our reasoning. See also **Point R1-6**.

**Point R1-6**  
P. 4, L 12: You introduce here the term bedrock topography, but do not specify what it means, especially as you further down use this quantity, to derive changes in sea level.

This section of the manuscript has been reorganised a bit, in order to define topography (T) in terms of sea level (B), which is a more intuitive definition. We refer more explicitly to Kendall et al. (2005), who follow the same definition on topography. We also define the ocean function explicitly (see also **Point R1-5**), which should help to better understand our reasoning. See also **Points R2-7 and R3-1**. Due to these
rearrangements, we have modified the beginning of Section 2.2 accordingly.

**Point R1-7**
P. 4, L 16: Here and in the following, you use the ‘cal’ symbol to specify variations with respect to a reference state. If so, you can of course reduce the number of equations, e.g., Eq.s 6, 11, 13 and 15 become redundant. Furthermore, you do not specify the reference state itself.

We understand the point, but we prefer to keep these equations as they stand, since they define very fundamental quantities that should be ‘lost’ if embedded in the text. We are now more explicit about the meaning of the reference state, also quoting the paper of Kendall et al. (2005), who uses the same approach to the SLE.

**Point R1-8**
P. 4, L 24: I don’t think that you have to refer to SSM19 to introduce the definition (7), but simply if follows from $\cal M / \int_e dA$.

We agree. The text has been modified accordingly.

**Point R1-9**
P. 5, L 2: This is somehow abrupt, and do you really need it here?

No, we do not need it here. We have removed the equation and a few lines of text.

**Point R1-10**
P. 5, L 4: What do you understand as a plausible surface load. At least you should guide the reader a bit more than referring to Bevis et al. 2016.

As we state, for plausible surface loads we mean surface loads that conserve the mass of ice+water. We better focus on Bevis et al. (2016) realization of plausible loads, by noting that the SLE is not relying upon a-priori definitions of the load distributed over the oceans, acting to compensate the ice load.
**Point R1-11**  
P. 5, L13: Is 'stem' used here synonymously to 'is associated'? If so, why then using a different word.

We use 'associated'.

**Point R1-12**  
P. 5, L 5ff: How do you ensure in (Eq. 9) that you considered all terms with respect to $\mathcal{M}$? In Eq. 5, I like that $\mathcal{O}$ can be one of \{-1, 0, 1\}.

In (previous) Eq. (9) we are confident we have included all the terms wrt $\mathcal{M}$, based on the analysis we perform in SSM19. We are not suspecting that anything is missing. Probably here the reviewer is referring, in his second observation, to Eq. 15 (not 5). In this case, we agree, $\mathcal{O}$ can (only) take the values -1, 0, and +1. We note that, in the revised manuscript.

**Point R1-13**  
P. 6, L 20ff: The definition of seal level from topography is a bit tricky to understand, as you did not define topography itself properly, also sea surface is not defined her properly. So, why not motivate it from Gauss definition of the geoid as that equipotential surface to which a static ocean surface would adjust? Then, the jump from N to $r^\prime ss$ is not as large. Also at the beginning you should specify what the sea level equation expresses. Only then you can obtain its most basic form as U-N. Also the relation to the water column might help the reader to understand its relation to loading more intuitively.

In **Point R1-6**, we have responded to the issue about the definition of topography, also with the aim of responding to similar points of **R2** and **R3**. In the revised manuscript, we have made a short and clear statement about the meaning of the SLE at the very beginning of this Section. We also make a reference to the water column just after having written the basic form of the SLE, to help intuition.
Point R1-14
P. 6, L 12: This is also known as Bruhns formula. As $\cal .$, is not applied to $\Phi$, I suggest to write this equation a bit different: $\cal G = \frac{\Phi - \Phi}{g_0}$, to remain in the used schema of formula symbols.

Yes, we use this notation, now, here and elsewhere. We quote the Bruns formula.

Point R1-15
P. 6, L 13: One important reference in your discussion of the SLE is missing which is Martinec and Hagedoorn, 2014, who discuss in detail the SLE of a rotating body. They, for instance, distinguish between gravity and gravitational potential, where the latter only considers the surface loading, i.e. variation due to surface mass change and solid earth deformation. The gravity potential, MH14 use for gravitational potential plus rotational effects. This is in accordance to Helmut Moritz, 1990, The figure of the earth.

So I would use gravity potential instead of geopotential.

Yes, we agree on all these points. We use the same terminology as in MH14, and we quote this important reference.

Point R1-16
P. 6, L 14: The meaning of $c$ only depending on $t$, can easily be explained simply by the fact, that the displacement of the reference potential surface, $N_0$, does not represent the potential level to which the water surface will adjust. This is not only due to mass conservation but also due to the changes in the ocean basin. So, it is a bit more complicate.

We believe that a nice and simple explanation of the existence and of the meaning of the ‘$c$ constant’ is given by Tamisiea (GJI, 2011), to which now we refer more explicitly.

Point R1-17
P. 7, L 10: Here and in the following, $O$ is dangerous to use, as it is easily mistaken for the reference state $0$.

It should not be the case, since the reference state label “0” is used as a subscript, while “o” in this equation and in the following is a superscript.

**Point R1-18**

P. 7, L 12: From type setting it is nicer to use text abbreviations in formulas with an unslanted font, e.g., ${\cal S}^\text{ave}$.

This is somewhat subjective, and since R2 and R3 did not make a similar observation, we keep all the equations as they stand.

**Point R1-19**

P. 7, 14: For the definition of the different sea-levels see below. For me, equivalent is more a renormed quantity, but here you are using it with respect to a model dependent quantity, depending on the current ocean area $A^o (\gamma, t)$ and $\mu (\gamma, t)$.

Both terms depend on the $\cal S$ and $\cal L^{[abc]}$.

This is another nice observation. We prefer to keep the abbreviation “equ” but it is clear that something must be said about the fact that $S^{\text{equ}}$ is associated to model-dependent “dynamic” quantities, like $A^o$ and $\cal O$.

We now mention this important point explicitly in the revised version of the paper.

**Point R1-20**

P. 7, L 8ff: This derivation is quite interesting, and I have needed some time to understand this alternative expression, especially why in Eq. 33 $T_0$ appears instead of $T$. May be, a specification what ‘using for $c$ the expression found in SSM19’ does mean, might help, as $c$ does not appear anymore. The interesting aspect is that this form is independent of the syphoning, which dominates $<S>^e$, and is put into $<G-U>^O$. May be this might help to understand Table 4 a bit more. So I guess the expression, $- <G-U>^O + S^\text{ave} = c$ is missing.
For that part of the question that concerns “c”, we mean that to obtain eq. (30) of the submitted manuscript we have used the expression obtained for c by the imposition of mass conservation, given in the supplement. We are now more explicit on the equation used for c, to avoid any ambiguities. We are not sure we have captured the meaning of the other issues raised in this point. In particular, we do not understand why the syphoning is mentioned in the Reviewer’ comment. So we ask for a clarification, if possible.

Point R1-21

P. 8 L 3: Wikipedia defines the ‘eustatic change’ as the alteration of sea level due to changes in volume of water or ice, and also due to changes in ocean topography, which goes back to Suess. The article to my point of view is based also on Rovere et al. (2016, https://doi.org/10.1007/s40641-016-0045-7), that is, also steric changes are considered. I would call Eq. 36, the equivalent sea level, as you also do at P. 10, L. 14ff, expressing an ocean equivalent of the ice, which is simply a renormation of water mass expressed as water height. Sˆave, I like. Your Sˆeus, I would call \bar{\cal S}ˆice, what is the equivalent of ice at current state. Your Sˆequ is a help quantity, I would keep its definition. FC76 phrase Eq. (36) to be the eustatic sea level although the ocean basin might slightly change. The authors are aware of the fact and “tried to avoid [. . .] the word (eustasy) has received so many qualifications since it was introduced by Suess”.
So, I would suggest not to use eustasy at all. It is a deprecated definition of sea level and mainly leads to misunderstanding.

We have re-worded and discussed in more detail the meaning of the definitions given, also quoting in particular the recent paper of Gregory et al. (2019) about the terminology to be used in the context of sea level change. We agree that misunderstanding could easily arise in this context, we hope the modified text can be considered satisfactory. We keep “eus” but we better specify his meaning, quoting Gregory et al. (2019) about the new term “barystatic”, which should be preferred to “eustatic”.
**Point R1-22**

P. 8, L 17: ‘essentially’ is a bit enigmatic, do you mean terms of d/o = 2/0 and 0/0 can be neglected as they are too small? See also MH2014.

Yes, it is what we mean. We revised the text to make this point clearer.

**Point R1-23**

P. 8. L 7: From here to the end of this section, you discuss the numerical implementation into SELEN. As the whole article is about SELEN, I would expect a bit more detail, how the equations are solved also with respect to the iterations you discuss in the following. So, this part I suggest to extend.

No, this part is not meant to discuss the numerical implementation into SELEN. As quoted in the text, details are given in SSM19 and we do not intend to duplicate that material here. We add a few lines, however, in which we briefly describe in a qualitative way how the solution is approached.

to A test run of SELEN

**Point R1-24**

In the introduction to this section you should mention that you also consider changes in rotation in addition to N, U and S.

Yes, right. We have done that.

**Point R1-25**

P. 10, L. 23: Why do you use an interpolation. To go from a fine to a coarse grid by interpolation is everytime a bit dangerous. I would expect here a filtering or a binning algorithm.

This is an important observation and we thank the Reviewer for pointing it out. Indeed, we obtained the pixelized topography through an interpolation by means of the GMT ‘grdtrack’ module. However, in our
numerical experiments, we used also binned topographies obtained through a sequence of ‘blockmean’ and ‘grdtrack’ GMT modules. The two realizations of topography turn out to be different in regions where small-scale relief features are present. However, we verified that all the relevant numerical results do not change within the numerical precision we used in the manuscript. In the revised text, we explicitly warn the user that for regional analyses focused on areas with small-scale topographic features, particular care shall be devoted to the realization of topography on the Tegmark grid.

Point R1-26

P. 11, L 10: 'Not agreed results on ve. Love numbers'. Do you have a reference for this statement, and why this is an argument to neglect compressibility?

We have rephrased (in LATEX) as <<\textbf{Since we \marginpar{R1-26} are not aware of published, community-agreed sets of Love numbers for a multi-layered compressible viscoelastic model}, in the test run we rest on \textbf{an incompressible profile, for which agreed results have been obtained \citep{spada2011benchmark}.}>> This is not an argument to neglect compressibility, it is just a remark. In the case Dr. Klemann \textbf{(R1)} can provide a reference contradicting this remark, we would change this statement accordingly. Beside this, SELEN4 can of course be used with any set of Love numbers, either community agreed or not agreed.

Point R1-27

P. 11, L 15: I would simply start the sentence: ‘Numerical values of [. . .] in Table 3 and its caption, respectively’, to avoid using two times reference, and the reader to search for the tidal love numbers in the table rows.

We agree; these changes have been done.

Point R1-28

P. 11, L 17 and P. 14, L 9: For the captions use ‘Glacial isostatic adjustment in the past’ and ‘[…] at present day’ or alternatively ‘Paleo glacial isostatic adjustment’ and ‘Present-day […]’.
We use “GIA at present” as a title for Section 3.3 and “GIA in the past” for Section 3.2.

**Point R1-29**
P 11, P 27: I would in one sentence introduce the three configurations you discuss in the following.

We have been more specific and careful to describe the three configurations.

**Point R1-30**
P. 13, L 2: Why do you mention the small region around Patagonia with a reference only. Why not citing Klemann et al. (2015, DOI 10.1007/s41063-015-0004-x) for the East Siberian shelf or Lambeck for the Sunda Shelf. Furthermore, the light blue areas may represent locations covered with grounded ice, but also floating ice might be present there. The ice extent is not shown in Fig. 5.

We are aware that Fig. 5 is not including the ice. This is clearly stated in the second paragraph of 3.2.2. Paleo-maps also showing the ice sheets shall be included in future releases of SELEN4. The reviewer is right, some important references were missing; now we quote Klemann et al and some others with reference to GIA modeling in specific areas. SELEN can visualize the Ocean Function, in addition to paleotopography. OF maps are located in the /OFU output folder, also showing the distribution of the floating and grounded ice at every time increment. To limit the length of the main text, we do not show them here, where we focus only on topography (folder /TOP). However we now mention them, so the reader is aware of these maps in SELEN. The whole paragraph has been somewhat rearranged to fit the requirements of R1.

**Point R1-31**
P. 13, L 20. As stated before the MH14 paper, is in my point of view an important contribution to the discussion of rotation in GIA.
We agree and we quote the MH14 paper also in this context.

**Point R1-32**
P. 13, L 25: ‘since’ used twice.

Right. We use ‘because’ the second time, instead of ‘since’.

**Point R1-33**
P. 14, L 8: For such statement, you should at least relate the order to that of other transient processes observed or proposed for rotational variations.

It is unclear to us, what the Reviewer is pointing to, here. We did not change the text.

**Point R1-34**
P. 14, L 11: Why ‘shall’, although you consider it.

Indeed. We now simply write “we consider”.

**Point R1-35**
P. 14, L 15: Here you define $\mathcal{S}$ as relative sea level, I suggest you to use this definition from the beginning, what also would clarify what the SLE is solving for.

We define $\mathcal{S}$ as “relative sea level change”, uniformly throughout the manuscript.

**Point R1-36**
P. 14, L 16: would repeat ‘frame’ also before b, c and d.

Done.

**Point R1-37**
P. 15, L 4: This is only clear from the fact that the syphoning effect is inside the definition of $\gamma$-depending part, otherwise the reader would wonder.

_We are not sure we understand this comment. We did not change the text._

**Point R1-38**
P. 15, L 8: Instead of ‘decontamination’, geodesists speak about ‘correction for’ an effect.

_We mention both terms._

**Point R1-39**
P 15, L 20: ‘then’ to ‘them’.

_OK._

**Point R1-40**
P. 16, L 23–25: This is an interesting aspect. Can you give a formula to indicate which term has to neglected as direct rotational effect?

_We now refer explicitly to Eq. S173 of the supplement, showing the 1+k structure that we quote in the main text. We do not think it is necessary to duplicate it here._

**Point R1-41**
P. 16, L 32: Is this the degree variance, and why not presenting the equation here?

_Because there is no necessity to duplicate Eq. S477 here, in our opinion._

to Conclusions

**Point R1-42**
P. 16, L 23–25: This is an interesting aspect. Can you give a formula to indicate which term has to neglected as direct rotational effect?

This point has been made above already, see R1-40.

**Point R1-43**
P. 16, L 32: Is this the degree variance, and why not presenting the equation here?

This point has been made above already, see R1-41.

**Point R1-44**
P. 17, L 12: replace ‘runs’ by ‘run’.

Yes, done.

**Point R1-45**
P. 17, L 15: Not sure, if one can speak about ‘physical realism’.

Right. Only realism.

to figures and tables

**Point R1-46**
Figure 1: In this color scheme, the dark ice thicknesses make it difficult to identify the coast lines. In this caption and in the further ones, I would not state which plotting command you used.

Right. We now use a red contour for the coastlines. We prefer to leave the command we use, for the sake of ease of reproducibility.

**Point R1-47**
Figure 2: I would skip this figure.
We have been tempted to skip this figure, too. But after consideration that it represents an important quantity, i.e., the final condition of the SLE, we would prefer to leave it.

**Point R1-48**

Table 2: ‘Density, [...] values specifying the parameters of the homogeneous layers defining the adopted [...]’. ‘Some spectral [...]’ I would skip this sentence, as it does not contribute to the table content.

We have simplified a bit the second sentence, which was effectively too long.

**Point R1-49**

Table 3: The PMTF is not defined in the manuscript. Without it, the A^[es] and [hkl]^T are not necessary. In this context it would be interesting, if you need also viscoelastic tidal love numbers.

We agree. We have skipped the PMTF information. We have left the tidal Love numbers information, as suggested by the Reviewer.

**Point R1-50**

Figure 8: Can you specify the extreme values reached in the respective plots?

Done.

**Point R1-51**

Table 4: I would skip the integrals over the whole earth, as these are clear from the definition of U and G.

We know that these are clear, but it is a nice indication of the precision of the SELEN4 numerical results. We prefer not to remove them.
Reply to comments of REVIEWER 2: Samuel Kachuck

Dear Editor, dear Dr. Kachuck,

Hereby we respond to the comments (in blue). In the annotated revised manuscript, the modifications made are marked in bold face and a label “R2-N” is found on the margin of the manuscript (where this is permitted by LaTeX), where R2 stands for Reviewer 3 and N is the point made by Reviewer 2 (labels are defined, in blue, in this letter).

We hope that we have responded satisfactorily to the constructive comments received, and we are looking forward to have your feedback.

Note that we have also corrected a few typos, added some references, improved the text in a few places, and re-edited some of the figures and tables.

Kind regards

Giorgio Spada & Daniele Melini


Interactive comment on “SELEN4 (SELEN version 4.0): a Fortran program for solving the gravitationally and topographically self-consistent Sea Level Equation in Glacial Isostatic Adjustment modeling” by Giorgio Spada and Daniele Melini

Comments by Reviewer: Samuel Kachuck

Point R2-0
This manuscript details the background and significant improvements to an opensource implementation of the sea-level equation. As solutions to this equation are essential to understanding relative sea level and geodetic evolution, the open-source method and documentation are welcome and the changes - including the physics of rotation and shoreline migration and the modularization of the code - warrant publication.
In addition, I believe the inclusion of the derivation to be useful to the community. Pursuant to that, there are areas where the derivation could be made even clearer, as noted below. Occasionally, a proliferation of variable definitions and abbreviations clouds the exposition, and a simplification would be welcome. We thank Samuel Kachuk for his positive evaluation and for the suggestions made. We have made efforts to address all his comments; when not, a justification is given. See the details given below.

**Point R2-1**
Finally, as the authors have just used this code’s predecessor to participate in the benchmarking exercises of Martinec et al. 2018, I think updating the contributed results with the new capabilities is warranted. In particular benchmark E, in which it seems they did not participate. As this is perhaps a more technical detail, I think it should be satisfactory to put this in the supplement, with perhaps a comment in the introduction to section 3. To this end, it would also be very interesting to see some discussion of how the included physics of rotation affect the results of those benchmarking cases.

A prototype version of SELEN4 has been used to compute results for the Martinec et al., 2018 exercise in all the considered benchmarks, including benchmark E (see Table 6 in Martinec et al., 2018). As suggested by R1, we explicitly state in the revised manuscript that the version of SELEN4 that we are publishing, when configured without rotation and in the same conditions of the 2018 benchmark, give numerical results that coincide with those published in that context. We believe that recomputing benchmark results with rotational effects would not add insight to the discussion, since all the 2018 exercises assumed no rotational feedback and therefore we cannot use them to validate our approach to rotational feedback modeling (of course, a new benchmarking initiative with a specific focus on the rotational effects would be of great interest). A discussion the difference between different rotation theories on the true polar wander is given in the manuscript, while for a more detailed analysis of the impact of rotational feedback on GIA fingerprints we now refer the
reader to a recently published paper (Spada and Melini, Water, 2019). See also response to point R1-1.

All further comments are presented in page-line format

Specific Comments

**Point R2-3**
1-15: I would like to mention that the python module, giapy, which was benchmarked in the Martinec, et al. 2018 paper (including shoreline migration and iteratively determined initial topography, though without rotation) is available for download, opensource (from https://github.com/skachuck/giapy), and similarly uses any normal-mode form love numbers. It is safe to say, however, that it has not yet been as comprehensively documented in its open-source release as SELEN.

We have modified the text accordingly, quoting giapy, in the abstract and in the introduction.

**Point R2-4**
2-15: “viscoelastic rheology...properly taken into account” see note to 1-15

Agree; see also point **R2-3**. We have been more specific, here.

**Point R2-5**
4-12: Could you use $S$ instead of $B$ to keep it all consistent?

No, we cannot, because we use S (in \texttt{cal} style) to denote the relative sea level change. B has a sense, here, because it recalls the term ‘bathymetry’.

**Point R2-6**
4-2: “ice and by the water in the ocean” refers to ice on the continent and water in the ocean, which is the basis for equation (2). Equation (1) refers to ice and water over the surface of the earth, i.e., $e$. 
We are not sure we can capture this point. Are the two equations in contradiction?

**Point R2-7**

4-13: given the detail of the surrounding derivation, could you spare a few words giving a precise definition of “topography”?

We define topography (T) in terms of sea level (B), which is a more intuitive definition. See also points R1-6 and R3-1, who have suggested a similar improvement.

**Point R2-8**

5-4: The meaning of “plausible” is not addressed in the referred paper beyond setting the integrated mass load variation, or equivalently the zero-degree coefficient, to zero.

A similar point has been made also by R1 (see R1-9 and R1-10). To respond to both, we have rephrased this part of the manuscript, and we are now more specific on the meaning of ‘plausible load’ and on the contents of the Bevis et al. paper. Thanks to both Reviewers for raising this issue.

**Point R2-9**

5-6: Could more interpretable labels be given to the components of the load variation than a, b, and c?

We have been thinking a lot to this opportunity, but we did not have any valid idea. Consider that labels (a, b, c) affect a number of other variables as the reviewer can see in the Supplement. And also the variable names that we use in the source code. Any suggestion about a more intelligent labeling is welcome.

**Point R2-10**

5-10: The repeated definition of the variation variables gets confusing. Could the font used throughout be defined as the variation with respect to some reference level?
On the contrary, we think that it becomes more clear, although redundant. We believe that we cannot omit these fundamental variables. See also our response to Point R1-7. To help the readers, we now say that we use calligraphic capital letters to denote variations of the corresponding fields.

**Point R2-11**
5-19: Can you give a physical meaning to the height variable $Q$? “Auxiliary variable” and “arbitrary reference density” are a little confusing and undermotivated. If you selected $\rho^r=\rho^w$, then it takes the form of a free water column, positive where there is liquid water, negative where the water is frozen, and zero otherwise.

The Reviewer is right, in the sense that other choices could probably provide a physical interpretation, which however could become too complicated and misleading. By Occam’s Razor, we prefer to leave this definition, leaving a more in-depth analysis of the possible physical meanings to a future study or to the readers.

**Point R2-12**
7-9: Throughout, when referring to equations in the supplement, could you refer to specific equations?

Revising the paper, we have done this in a number of places to respond to R1. These equations/sections are marked by Sx or Sx.y.

**Point R2-13**
9-12: Could you briefly describe the significance of the external and internal iterations?

The same was essentially requested in R2-23, to which we have responded by adding a new short paragraph in which we summarise the meaning of external and internal iterations.

**Point R2-14**
16-12: You could cite somewhere the work done by Barletta and Bordoni (2013) on the effects of implementation of ice histories.

*We would be happy to cite it, but we do not see in which part of our manuscript this could be useful. Any suggestions?*

**Point R2-15**

17-27: see note to 1-15.

*Yes, we agree. The text has been modified accordingly.*

**Point R2-16**

Table 2: For clarity, you might articulate that these are piecewise constant layers whose upper radii are given by the variable $r$.

*We have modified the style of the Table, giving both the lower and the upper radius, which makes things clearer, in our opinion. Note that the rigidities in the lower mantle where erroneously multiplied by a spurious factor of 10. This has been fixed.*

**Point R2-17**

Figure 4: Is the ordering significant? If so, state what guides it. The non-numerical ordering makes it difficult to locate them from the text quickly.

*The reviewer is definitively right here. Form the numbers, one cannot ‘see’ where the location is. We have added a map to help the reader. Thanks.*

**Point R2-18**

Figures 5 and 6: the color scale looks different between these figures for the resolutions in the manuscript draft. Make sure to check this as it goes through editorial.

*Thanks, we shall pay attention to this.*

Technical corrections
Point R2-19
2-2: “available since” is a bit awkward, consider “dating from”

OK.

Point R2-20
2-10: “Despite GIA is now tightly” missing clause between “Despite” and “GIA”

We have rephrased into “Despite the GIA phenomenon is now tightly...”; we hope that this can be considered correct.

Point R2-21
2-16: “limited the” missing word “to”

The whole paragraph has been rephrased in response to Reviewer R1.

Point R2-22
2-16: “elastic rheology” to “elastic deformation”

Yes, we have changed this, also in response to R1-2.

Point R2-23
2-20: “Love numbers” to “Love number”

OK.

Point R2-24
2-28: “is hosting” to “has hosted” for tense agreement

OK.

Point R2-25
2-30: “Since year” missing work “the”

OK.
Point R2-26
3-24: “taken by the SLE” to “the SLE takes”

OK.

Point R2-27
3-24: remove “an”

OK.

Point R2-28
6-6: Should $\hat{s}s$ be $\hat{s}e$?

Yes, indeed. A Major typo that has been noted also by others.

Point R2-29
8-25: “accomplished projecting” missing word “by”

Yes.

Point R2-30
11-14: “with origin in the whole Earth’s: : :” to “with the origin at the Earth’s: : :”

Yes.

Point R2-31
12-7: “from high-resolution but also from” to “from both high-resolution and”

Yes.

Point R2-32
12-21: The first clause “Differently...program” is awkward to read and the sentence would be fine without it.
Agree, 1st clause removed.

**Point R2-33**
13-5: The meaning of the word “safely” is not clear.

Agree, word ‘safely' removed.

**Point R2-34**
13-25: “since” used twice in one sentence.

Agree, one of the two ‘since’ has been substituted by ‘because’. See also R1-32.

**Point R2-35**
13-27: Consider adding “and shown in Figure 7” at the end of that sentence for a smoother transition.

Agree on the smooth transition.

**Point R2-36**
13-30: The sentence beginning with “Further results” could be revised to “Dashed curves show results obtained with the traditional theory.”

Much better.

**Point R2-37**
14-9: Inconsistent abbreviations in section headings; “GIA versus “Glaical Isostatic Adjustment”

See also point R1-28, we are now consistent with the use of <<GIA>> in the section headings.

**Point R2-38**
14-25, 14-31: “regardless the” missing word “of”

OK.
**Point R2-39**  
15-3: “very affected by” to “sensitive to”  
OK.

**Point R2-40**  
16-8: “Disclosing” to “Tracing”  
OK.

**Point R2-41**  
17-6: “which has been” to “which was”  
OK.

**Point R2-42**  
17-14: move “in SELEN” to the end of the sentence  
OK.

**Point R2-43**  
17-15: “increased physical realism” is a little odd  
Yes, we change into realism, see also R1-45.

**Point R2-44**  
Figure 7: label ‘x’ and ‘y’ in the right-hand panel.  
OK.
Reply to comments of REVIEWER 3: Geruo A

Dear Editor, dear Dr. A,

Hereby we respond to the comments (in blue). In the annotated revised manuscript, the modifications made are marked in bold face and a label “R3-N” is found on the margin of the manuscript (where this is permitted by LaTeX), where R3 stands for Reviewer 3 and N is the point made by Reviewer 3 (labels are defined, in blue, in this letter).

We hope that we have responded satisfactorily to the constructive comments received, and we are looking forward to have your feedback.

Note that we have also corrected a few typos, added some references, improved the text in a few places, and re-edited some of the figures and tables.

Kind regards

Giorgio Spada & Daniele Melini


Interactive comment on “SELEN4 (SELEN version 4.0): a Fortran program for solving the gravitationally and topographically self-consistent Sea Level Equation in Glacial Isostatic Adjustment modeling” by Giorgio Spada and Daniele Melini

Comments by Reviewer: Geruo A

**Point R3-0**

The manuscript describes an updated numerical model that solves the sea level equation. Compared with its previous iterations, the new model now accounts for shoreline migration and rotational feedback, and it features enhanced portability and computational efficiency. The theory session is generally easy to follow. The result session provides a clear overview of the model configuration (i.e. ice and Earth model inputs) and highlights the newly implemented features (i.e. shoreline migration and
rotational effects). The manuscript fits the scope of the journal and it is generally well written. I have a few minor comments listed as follows.

We thank Geruo A for his positive evaluation and for the suggestions made. We have made efforts to address all his comments; when not, a justification is given. See the details given below.

**Point R3-1**
The authors first define the sea level as $B = -T$ (Eq. 3 and Page 3, Line 21), and later express it in Eq. 16 as the difference between the sea surface height and the height of the solid Earth surface. I think Eq. 16 is a more intuitive definition of the sea level. Based on this equation, it is also straightforward to define topography as $T = -B$. I suggest the authors introducing Eq. 16 before Eq. 2. Following this definition, the ocean function can also be defined immediately ($O = 1$ when $B > 0$ with no grounded ice).

This section of the manuscript has been rephrased, in order to define topography ($T$) in terms of sea level ($B$), as also suggested by Reviewer 1. See also Points [R1-6](#) and [R2-7](#).

**Point R3-2**
Figure 8. Please consider increasing the range of the color scale to make the plot less saturated.

While the two top panels are a bit saturated, the two bottom ones are not (see min/max values in the caption). Using two different scales for the top and bottom hinder a easy intercomparison between the fingerprints, so we have decided to leave the figure as it stands.

**Point R3-3**
Page 15, Line 15 to the end of session 3.3.2. It would be helpful to clarify the typical range of errors of tide gauge measurement. I think this would help the readers understand the significance of the difference among the reported model runs.
This is a very useful suggestion, and we thank the Reviewer for that. We make this point when we compare the ‘best’ SELEN prediction (R100/L512/I5) to the original implementation of ICE-6G\_C(VM5a).

**Point R3-4**
Table 5. The label and the caption of Column (e) should be ICE-6G instead of ICE-5G.

Yes. We have made this change.

**Point R3-5**
Page 16, Line 1. Please clarify how the coherence between the two predictions is quantified.

For coherence we mean that the two predictions have the same sign. We avoid using this term and rephrase, now.

**Point R3-6**
Page 16, Line 20. It would be helpful to clarify that the “direct effect” is associated with the change in centrifugal potential. This would help the readers follow the discussion at Lines 21-25.

Right, OK.

**Point R3-7**
Page 17, Lines 19-25. This part of the conclusion focuses on the computational aspect of the model while the current result session is not organized in a way to highlight this aspect. It would be helpful to include a brief summary in the result session to justify this part of the conclusion, especially regarding the second point at Line 21.

In this part of the conclusions, we just want to briefly highlight the major improvements we implemented in SELEN4 from a technical and practical standpoint. We think that it is not worth discussing technical aspects like code organization or customization of input files in the main paper, since these are illustrated in detail in the user guide. On the other hand,
we agree with the Reviewer that our statement about code parallelism on line 21 needs some quantitative support, so now we explicitly refer to the supporting material, where the scaling of SELEN4 has been thoroughly characterized.
SELEN⁴ (SELEN version 4.0): a Fortran program for solving the gravitationally and topographically self-consistent Sea Level Equation in Glacial Isostatic Adjustment modeling

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Abstract. We present SELEN⁴ (a SealEveL EquatioN solver), an open-source program written in Fortran 90 that simulates the Glacial Isostatic Adjustment (GIA) process in response to the melting of the late-Pleistocene ice sheets. Using a pseudo-spectral approach complemented by a spatial discretization on an icosahedron-based spherical geodesic grid, SELEN⁴ solves a generalised “Sea Level Equation” (SLE) for a spherically symmetric Earth with linear viscoelastic rheology, taking the migration of the shorelines and the rotational feedback on sea level into account. The approach is gravitationally and topographically self-consistent, since it considers the gravitational interactions between the solid Earth, the cryosphere and the oceans, and it accounts for the evolution of the Earth’s topography in response to changes in sea level. Program SELEN⁴ can be employed to study a broad range of geophysical effects of GIA, including past relative sea-level variations induced by the melting of the late-Pleistocene ice sheets, the time evolution of paleogeography and of the ocean function since the Last Glacial Maximum, the history of the Earth’s rotational variations, present-day geodetic signals observed by Global Navigation Satellite Systems and gravity field variations detected by satellite gravity missions like GRACE (the Gravity Recovery and Climate Experiment). The GIA fingerprints constitute a standard output of SELEN⁴. Along with the source code, we provide a supplementary document with a full account of the theory, some numerical results obtained from a standard run, and a User guide. Originally, program SELEN was conceived by GS in 2005 as a tool for students eager to learn about GIA, and it has been the first SLE solver made available to the community.

1 Introduction

In the last few decades, GIA modeling has progressively gained a central role in the study of contemporary sea-level change. Sea-level variations observed at the tide gauges deployed along the world coastlines need to be corrected for the effect of GIA to enlighten the impact of global warming. As discussed in the review of Spada and Galassi (2012), a precise estimate of global sea-level rise has been possible only after Peltier and Tushingham (1989) first solved the SLE using an appropriate spatial resolution, building upon the seminal papers of Farrell and Clark (1976) and Clark et al. (1978). Since then, a number of GIA
models characterised by different assumptions about the Earth’s rheological profile and the history of the late-Pleistocene ice sheets have been proposed, constrained by sea-level proxies dating since the Last Glacial Maximum (LGM, 21,000 years ago). For a review of the development of GIA modelling, the reader is referred to Whitehouse (2009), Spada (2017) and Whitehouse (2018). GIA models have provided increasingly accurate estimates of global mean secular sea-level rise (a summary is given in Table 1 of Spada and Galassi, 2012), but have also the potential of describing the patterns of future trends of sea level in a global change scenario (see e.g., Bamber et al., 2009; Spada et al., 2013). Since the beginning of the “altimetry era” (1992-today) and the launch of the Gravity Recovery and Climate Experiment (GRACE, see Wahr et al., 1998) in 2002, GIA modeling has re-gained momentum, providing the tools for isolating the effects of global warming i) from absolute sea-level data (Nerem et al., 2010; Cazenave and Llovel, 2010) and ii) from the Stokes coefficients of the gravity field (see Leuliette and Miller, 2009; Cazenave et al., 2009; Chambers et al., 2010; WCRP, 2018) to infer the ocean mass variation. Despite the GIA phenomenon is now tightly integrated into the science of global change (Church et al., 2013b), little efforts have been paid so far to the development of open-source codes for the solution of the SLE, although several Post Glacial Rebound simulators (like e.g., TABOO, see Spada et al., 2004, 2011) and Love numbers calculators have been made available to the community (Spada, 2008; Melini et al., 2015; Bevis et al., 2016; Kachuck and Cathles, 2019). As far as we know, the only publicly available and open-source program in which the SLE is solved in its complete form is SELEN. The SLE solver ISSM-SESA v1.0 of Adhikari et al. (2016), being oriented to short term cryosphere and climate changes, only accounts for the elastic deformation of the Earth. The open source SLE solver giapy (Kachuck, 2017; Martinec et al., 2018), available from https://github.com/skachuck/giapy, can deal with complex ice models and viscoelastic rheology but it is not taking rotational effects into account.

SELEN was first presented to the GIA modeling community by Spada and Stocchi (2007), who numerically implemented the SLE theory reviewed in Spada and Stocchi (2006). SELEN was fully based on the classical formulation of Farrell and Clark (1976); hence, the fixed-shorelines approximation was assumed, and no account was given of rotational effects on sea-level variations. SELEN used the Love number calculator TABOO (see Spada et al., 2011) as a subroutine and was tied to the Generic Mapping Tools (GMT, see Wessel and Smith, 1998) for the construction of the present-day ocean function. In SELEN and in all its subsequent versions, the numerical integration of the SLE over the sphere takes advantage of the icosahedron-based pixelization by Tegmark (1996). Similarly, all the versions are based upon the pseudo-spectral method of Mitrovica and Peltier (1991) and Mitrovica et al. (1994) for the solution of the SLE. Originally, SELEN came without a User guide, and it was disseminated via email by the authors. After SELEN was first published in 2007, a number of improvements were made in terms of computational efficiency, portability and versatility, but leaving the physical ingredients of the original code unaltered. This led to a new version of the program, named SELEN 2.9 and announced by Spada et al. (2012). Since 2015, the Computational Infrastructure for Geodynamics (CIG, http://www.geodynamics.org/) has hosted SELEN 2.9 in its Short-term Crustal Dynamics section (see http://geodynamics.org/cig/software/selen), from where it can be freely downloaded along with a theory booklet and a fully detailed User guide (Spada and Melini, 2015). Since the year 2012, with the aid of Florence Colleoni and thanks to the feedback of a number of colleagues and students, GS and DM have implemented new modules aimed at solving the SLE in the presence of rotational effects and taking the migration of the shorelines into account. This has progressively led
to several interim versions of the program (SELEN 3.x), which have been tested intensively and validated during the years, but never officially released. We note that building upon SELEN, some colleagues have independently developed other versions of the code aimed at specific tasks, such as the study of the coupling between the SLE and ice dynamics (de Boer et al., 2017).

Taking advantage of the experience developed since SELEN was first designed, we are now publishing a new version of the code named SELEN\(^4\) (SELEN version 4.0). With respect to previous versions, SELEN\(^4\) has been improved in several aspects. \(i\) The underlying SLE theory has been fully revised and now accounts both for horizontal migration of shorelines and for rotational effects, resulting in a more realistic description of the GIA processes. \(ii\) The package has been streamlined and reorganized into two independent modules: a solver, which obtains a numerical solution of the SLE in the spectral domain, and a post-processor, which computes a full suite of observable quantities through a spherical harmonic synthesis. This new structure facilitates code portability, reusability and customization, enabling the adaptation of SELEN\(^4\) to new use-cases. \(iii\) The SELEN\(^4\) modules have been completely rewritten using symbol names that are closely matching those of the variables introduced in this paper, for the ease of code readability. Particular attention has been paid to the optimization of the SLE solver, resulting in a large extent of shared-memory parallelism, which allows for an efficient scaling to high resolutions on multi-core systems. \(iv\) SELEN\(^4\) has been decoupled from the GMT software package, which is no longer strictly required to run a GIA simulation, thus facilitating code portability on high-performance systems where GMT may not be available.\(^{15}\) SELEN\(^4\) still takes advantage of GMT (version 4) to produce various graphical outputs through plotting scripts included in the distribution package. \(v\) SELEN\(^4\) no longer calls the post glacial rebound solver TABOO as an internal subroutine to compute the viscoelastic loading and tidal Love numbers, which are instead supplied by the user through a data file. In this way, any set of Love numbers can be used in SELEN\(^4\), possibly overcoming some of the intrinsic limitations of existing Love numbers calculators like TABOO. \(vi\) Recently, a prototype version of SELEN\(^4\) has been successfully validated in a community benchmark of independently developed GIA codes (Martinec et al., 2018). In the benchmark tests, simplified surface loads have been employed, and the rotational effects have been ignored. In a further benchmark effort, we are planning to adopt a realistic description of the surface load, and the contribution from Earth rotation.

The paper is organized into three main sections. In Section 2, we present a condensed theory background for the SLE. In Section 3, we describe the outputs obtained by a standard, intermediate-resolution run of SELEN\(^4\). In Section 4, we draw our conclusions.

\section{Theory}

Here we obtain the sea-level equation (SLE) from first principles, leaving a number of details of the theory to the supplementary material, hereafter referred to as SSM19. The focus is on the various forms the SLE takes, which are characterised by increasing complexity; the goal is to obtain a formulation suitable for a numerical discretisation, which is given and analysed in SSM19 and implemented in SELEN\(^4\). In an attempt to simplify the presentation, and to obtain compact expressions for all the quantities involved, we are not exactly following the traditional notation adopted in the literature since the seminal paper by Farrell and Clark (1976), hereafter referred to as FC76. The number of definitions and variables involved in the construction of
the SLE is remarkable and some derivations are cumbersome; to facilitate the readers – especially those who are approaching the GIA problem for the first time – we refer to the synopsis presented in SSM19 (see Table S8). Program SELEN4 is written in a plain way, adopting constants and variables names that follow the same notation employed in this theory section and in SSM19, with the aim of facilitating code readability.

2.1 Surface loads

We consider the system composed by the ice and by the water in the oceans, at a given time \( t \). Its mass can be expressed as

\[
M(t) = \int L dA, 
\]

where \( L(\gamma, t) \) is the surface load, \( \gamma \) stands for \((\theta, \lambda)\) where \( \theta \) and \( \lambda \) are the geocentric colatitude and longitude, respectively, the integral is over the whole Earth’s surface, and \( dA = a^2 \sin \theta d\theta d\lambda \) is the area element, \( a \) being the average radius of the Earth. According to Eq. (1), \( L \) represents the mass per unit area distributed over the Earth’s surface:

\[
L(\gamma, t) = \frac{dM}{dA}, 
\]

with \( M(t) = M^i + M^w \), where \( M^i \) and \( M^w \) are the mass of the ice and of the water at time \( t \). By obtaining explicit expressions for \( M^i \) and \( M^w \), in SSM19 we show that \( L \) can be expressed as the sum of two contributions, which account for the load exerted by the grounded ice and for the load of water on the oceans floors, respectively:

\[
L(\gamma, t) = \rho^i IC + \rho^w BO, 
\]

where \( \rho^i \) and \( \rho^w \) are the ice and ocean water densities, \( I \) is the ice thickness, \( B \) is sea level (i.e., the offset between the sea surface and the surface of the solid Earth), \( C = 1 − O \) is the continent function (CF), where the ocean function (OF) is

\[
O(\gamma, t) \equiv \begin{cases} 
1, & \text{if} \quad T + \frac{\rho^i}{\rho^w} I < 0 \\
0, & \text{if} \quad T + \frac{\rho^i}{\rho^w} I \geq 0,
\end{cases} 
\]

where following Kendall et al. (2005) (see their Eq. 2) we have defined bedrock topography \( T \) as the inverse of the globally defined sea level

\[
T(\gamma, t) = -B. 
\]

We remark that due to the horizontal migration of the shorelines and to the transition between floating and grounded ice, \( O \) and \( C \) are, in general, time-dependent.

In the following, we are concerned with time variations of the fields involved in the SLE, which are denoted by calligraphic capital letters. These variations are relative to a reference equilibrium state, established for \( t \leq t_0 = 0 \), during which all fields have a constant value. Conventionally, time \( t = t_0 \) denotes the time at which the surface load changes
Accordingly, using Eq. (3) in (1), the mass variation $\mathcal{M}(t) = M - M_0$ of the system composed by ice and water is

$$\mathcal{M}(t) = \rho^i \int_e (IC - I_0C_0) \, dA + \rho^w \int_e (BO - B_0O_0) \, dA,$$

where subscript $0$ denotes reference quantities (i.e., constant values attained for $t \leq 0$), and the first term on the right hand side represents the mass variation of the grounded ice, which we denote by $\mu(t)$. Since mass must be conserved, we have

$$\mathcal{M}(t) = \int_e \mathcal{L} \, dA = 0,$$

where

$$\mathcal{L}(\gamma, t) \equiv L - L_0,$$

is the surface load variation. From Eq. (7) it follows immediately that an equivalent form of the mass conservation constraint is

$$\langle \mathcal{L} \rangle^e(t) = 0$$

where $\langle \ldots \rangle^e$ indicates the average over the whole Earth surface. Hereinafter, we shall consider only plausible surface loads, that we define here as those loads for which the mass is conserved. A discussion on how these surface loads can be realized is given by Bevis et al. (2016), who however have adopted ad hoc assumptions on how the mass lost from the ice sheets is redistribute. The SLE, however, does not rely upon such assumptions, since it defines a gravitationally self-consistent and non-uniform mass distribution over the oceans.

In SSM19, a suitable decomposition is found for $\mathcal{L}$, namely

$$\mathcal{L}(\gamma, t) = \mathcal{L}^a + \mathcal{L}^b + \mathcal{L}^c,$$

where the first term

$$\mathcal{L}^a(\gamma, t) = \rho^i C I$$

is associated with the ice thickness variation

$$I(\gamma, t) = I - I_0,$$

the second

$$\mathcal{L}^b(\gamma, t) = \rho^w O S$$

is associated with sea-level change

$$S(\gamma, t) = B - B_0,$$
and the third
\[ \mathcal{L}^c(\gamma, t) = \rho^r Q \mathcal{O}, \]  
(15)
is associated with \textit{OF variations}
\[ \mathcal{O}(\gamma, t) = O - O_0, \]  
(16)
where \( \rho^r \) is an arbitrary reference density and \( Q \) a time invariant auxiliary variable. \textbf{We note that, by the same definition of OF, its variation \( \mathcal{O}(\gamma, t) \) can only take the values \((-1, 0, +1)\).}

2.2 The Sea Level Equation

The Sea Level Equation expresses the relationship between sea-level change, the sea surface variation, and the vertical displacement of the solid Earth. In this section, various forms of this relationship are described.

Above, \textit{sea level} \( B \) has been qualitatively defined as the offset between the sea surface and the surface of the solid Earth. More specifically, denoting by \( r^{ss}(\gamma, t) \) and \( r^{se}(\gamma, t) \) the radii of the sea surface and of the Earth’s solid surface in a geocentric reference frame, respectively, sea level can be equivalently expressed as
\[ B(\gamma, t) = r^{ss} - r^{se}, \]  
(17)
and in the reference state
\[ B_0(\gamma) = r_0^{ss} - r_0^{se}. \]  
(18)

So, introducing the \textit{sea surface variation}
\[ \mathcal{N}(\gamma, t) = r^{ss} - r_0^{ss}, \]  
(19)
and the \textit{vertical displacement} of the solid surface of the Earth
\[ \mathcal{U}(\gamma, t) = r^{se} - r_0^{se}, \]  
(20)
using Eq. (14), we obtain the SLE in its most basic form
\[ S(\gamma, t) = \mathcal{N} - \mathcal{U}, \]  
(21)
expressing the variation of the height of the water column bounded by the sea surface and the ocean floor.

The sea surface variation is tightly associated with variations in the Earth’s gravity field. Indeed, FC76 have shown that
\[ \mathcal{N}(\gamma, t) = \mathcal{G} + c, \]  
(22)
where the \textit{displacement of the geoid} is obtained by Bruns formula
\[ \mathcal{G}(\gamma, t) = \frac{\Phi - \Phi_0}{g}, \]  
(23)
(Heiskanen and Moritz, 1967), where \( \Phi \) is gravity potential, which includes both the effects of surface loading and those from rotational variations (Martinec and Hagedoorn, 2014). \( g \) is the reference gravity acceleration at the Earth’s surface, \( c \) is a spatially invariant term introduced by FC76 to ensure mass conservation. The \( c \) term is known within the GIA community as the FC76 constant, and its physical origin is explained in detail by Tamisiea (2011). Note that as a consequence of Eq. (22), the change in the sea surface elevation does not coincide with the variation of the geoid elevation. Hence, using (22), the SLE (21) becomes

\[
S(\gamma, t) = \mathcal{R} + c, 
\]

where

\[
\mathcal{R}(\gamma, t) = \mathcal{G} - \mathcal{U}, 
\]

shall be referred to as sea-level response function.

We now assume that the responses to surface loading and to changes in the centrifugal potential can be combined linearly. Accordingly, the SLE (24) can be further rearranged as

\[
S(\gamma, t) = \mathcal{R}^{\text{sur}} + c + \mathcal{R}^{\text{rot}}, 
\]

where

\[
\mathcal{R}^{\text{sur}}(\gamma, t) = \mathcal{G}^{\text{sur}} - \mathcal{U}^{\text{sur}} 
\]

and

\[
\mathcal{R}^{\text{rot}}(\gamma, t) = \mathcal{G}^{\text{rot}} - \mathcal{U}^{\text{rot}} 
\]

are the surface and the rotation sea-level response functions, whereas

\[
\mathcal{G}(\gamma, t) = \mathcal{G}^{\text{sur}} + \mathcal{G}^{\text{rot}} 
\]

and

\[
\mathcal{U}(\gamma, t) = \mathcal{U}^{\text{sur}} + \mathcal{U}^{\text{rot}} 
\]

are the geoid and the vertical displacement response functions, respectively.

By the constraint of mass conservation given by Eq. (7), the FC76 constant is easily determined. Using the expression found in SSM19 (see, in particular, Eq. S52), the SLE (26) becomes

\[
S(\gamma, t) = S^{\text{ave}} + (\mathcal{R}^{\text{sur}} - < \mathcal{R}^{\text{sur}} >^o) + (\mathcal{R}^{\text{rot}} - < \mathcal{R}^{\text{rot}} >^o), 
\]

where \(< \ldots >^o\) indicates the average over the (time-dependent) ocean surface defined by \( O = 1 \), and

\[
S^{\text{ave}}(t) = S^{\text{equ}} + S^{\text{ofu}}, 
\]
where $S^{equ}$ and $S^{ofu}$ are two spatially invariant terms. The first, referred to as "equivalent sea-level change", is

$$S^{equ}(t) = -\frac{\mu}{\rho w A^o},$$

where $\mu$ is the mass variation of the grounded ice and $A^o$ is the area of the oceans. The second

$$S^{ofu}(t) = \frac{1}{A^o} \int T_0 O dA$$

depends explicitly upon variations of the OF, either due to the horizontal migration of the shorelines or to transitions from grounded to floating ice (or vice versa). It is important to note that $S^{equ}(t)$ and $S^{ofu}(t)$ are both model-dependent, via $A^o$ and $O$. Evaluating the ocean-average of both sides of Eq. (31), and observing that $\langle \langle R^o \rangle^o \rangle = \langle R \rangle^o$ and that $\langle S^{ave} \rangle = S^{ave}$, it is easily verified that $S^{ave}$ simply represents the ocean-averaged relative sea-level change

$$S^{ave}(t) = \langle S >^o. \tag{35}$$

In consequence of that, from Eq. (31) we see that the regional imprint of GIA on relative sea-level change is totally determined by the response functions $R^{sur}$ and $R^{rot}$.

In the classical FC76 framework, a constant OF is assumed and the effects arising from Earth rotation are neglected. Hence $R^{rot} = 0$ and $O = 0$, with the latter condition implying $S^{ofu} = 0$ because of (34). In this context, the SLE simply reduces to

$$S^{FC76}(\gamma,t) = S^{eus} + (R^{sur} - \langle R^{sur} >^o), \tag{36}$$

where

$$S^{eus}(t) = -\frac{\mu}{\rho w A^{op}} \tag{37}$$

is often referred to as eustatic sea-level change, and $A^{op}$ represents the present-day area of the surface of the oceans. Hence, for a rigid and non-gravitating Earth (for which $R^{sur} = 0$), $S^{eus}$ would represent the (spatially invariant) relative sea-level change. Note that $S^{eus}$, which only depends on the history of the past grounded ice volume (Milne and Mitrovica, 2008), should not be confused with $S^{ave}$ given by Eq. (32) since the latter is dynamically dependent upon the Earth’s response through $A^o(t)$ and $O(t)$ (e.g., Spada, 2017). We note that in the current terminology for sea level (Gregory et al., 2019) the term barystatic should be preferred to eustatic, a term originally attributed to Suess (1906) and that since then has received many qualifications (see FC76, page 648).

Following e.g., Milne and Mitrovica (1998), the surface response function $R^{sur}$ in (31) is obtained by a 3-D spatio-temporal convolution

$$R^{sur}(\gamma,t) = \Gamma^s \otimes L, \tag{38}$$

where $\Gamma^s(\gamma,t)$ is the surface sea-level Green’s function. The details of the expansion of $R^{sur}$ in series of spherical harmonics are somewhat cumbersome and can be found in SSM19. Here we only note that using Eq. (10) we have $R^{sur} = R^a + R^b + R^c$. 

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where the three terms are obtained by convolving $\Gamma_s$ with $L^a$, $L^b$ and $L^c$, respectively. Contrary to $R^{sur}$, the harmonic coefficients of the rotational response $R^{rot}(\gamma, t)$ are directly obtained by a 1-D time convolution

$$R^{rot}_{lm}(t) = \Upsilon^s_{l} \ast \Lambda_{lm},$$

(39)

where $\Upsilon^s_{l}(t)$ is the rotation sea-level Green’s function and $\Lambda_{lm}(t)$ are the coefficients of degree $l$ and order $m$ of the expansion of the variation of centrifugal potential $\Lambda(\gamma, t)$ associated with changes in the Earth’s angular velocity (Milne and Mitrovica, 1998). In SSM19, it is shown that $\Lambda(\gamma, t)$ is essentially a spherical harmonic function of degree $l = 2$ and order $m = \pm 1$, being the other degree $l = 2$ terms and the degree $l = 0$ in the expansion of $\Lambda(\gamma, t)$ negligible (see also Martinec and Hagedoorn 2014).

Thus, the SLE (31) can be rearranged as

$$S(\gamma, t) = S^{ave} + R^a + R^b + R^c + R^{rot},$$

(40)

where the primed response functions are

$$R^{abc}_{\gamma}(\gamma, t) = R^{abc} - <R^{abc}>,$$

(41)

$$R^{rot}_{\gamma}(\gamma, t) = R^{rot} - <R^{rot}>,$$

(42)

We note that $R^b$ depends on $OS$ through the surface load variation $L^b$ (see Eq. 13); in SSM19 it is shown that this holds for $R^{rot}$ as well. Following Mitrovica and Peltier (1991), in view of the numerical solution of the SLE it is therefore convenient to transform Eq. (40) in such a way that $OS$ becomes the unknown in lieu of $S$. This is accomplished by projecting Eq. (40) on the OF (i.e., multiplying both sides of the SLE by $O$), which provides the final form of the SLE

$$Z(\gamma, t) = Z^{ave} + K^a + K^b(Z) + K^c + K^{rot}(Z),$$

(43)

where

$$Z(\gamma, t) = OS,$$

(44)

and

$$Z^{ave}(\gamma, t) = OS^{ave} = O\left(S^{ave} + S^{off}\right),$$

(45)

$$K^{abc}(\gamma, t) = OR^{abc},$$

(46)

$$K^{rot}(\gamma, t) = OR^{rot}.$$  

(47)

The dependence of $K^b$ and $K^{rot}$ upon $Z$ in Eq. (43) manifests the implicit nature of the SLE, which is a 3-D non-linear integral equation, similar, in some respect, to an inhomogeneous and non linear Fredholm equation of the second kind (e.g., Jerri, 1999; Spada, 2017). For the spectral discretization of Eq. (43) and for a detailed illustration of the scheme adopted to solve the SLE, the reader is referred to sections S7 and S8.7 of SSM19. Here it is only worth to mention that the SLE is
solved by a pseudo-spectral iterative approach (Mitrovica and Peltier, 1991; Mitrovica and Milne, 2003) complemented by a spatial discretization on an icosahedron-based spherical geodesic grid (Tegmark, 1996). Two nested iterations are adopted, where the external one refines progressively the topography using the solution of the SLE obtained in the internal one.

3 A test run with SELEN⁴

In the following, we illustrate some of the outputs of a standard SELEN⁴ run in which the resolution of the Tegmark grid is set to \( R = 44 \) (see S8.6) while the maximum harmonic degree of the spectral decomposition is \( l_{\text{max}} = 128 \). Note that condition \( P \geq \frac{l_{\text{max}}^2}{3} \), where \( P = 40R(R-1) + 12 \) is the number of pixels in the grid for a given \( R \) value, must be necessarily met to preserve the properties of the spherical harmonics on the grid as discussed in Section S8.6 and in Tegmark (1996). In the test run, we solve the SLE by three external and three internal iterations \( (n_{\text{ext}} = n_{\text{int}} = 3) \), as customarily adopted in GIA studies as in Kendall et al. (2005). Henceforth, the notation \( R44/L128/I3 \) shall be employed to denote these fundamental SELEN⁴ settings. Of course, with increasing values of parameters \( (R, l_{\text{max}}, n_{\text{ext}}, n_{\text{int}}) \), more accurate results are expected, which however might come with a substantial increase in the computational burden. In the SELEN⁴ test run we account for both loading and rotational effects.

We assume that the user has installed and executed the program following the indications given in the User guide of SELEN⁴. Most of the program outputs discussed in this section have been obtained using the same configuration file that comes with the SELEN⁴ package. However, some results shall be based on different settings, in order to appreciate the sensitivity of the outputs on key configuration parameters. We first describe the GIA model adopted in the test run, which consists of three elements, i.e., an ice melting history, a description of the present-day global relief, and a 1-D rheological model of the Earth’s mantle. Then, browsing the output folders of SELEN⁴, we illustrate and discuss two distinct output sets, pertaining to the past and to the present effects of GIA on sea-level change and on geodetic variations, respectively. The features of the test run are summarized in Table 1. For reference, on a 12-core Mid-2012 Mac Pro, the execution time of this test run of SELEN⁴ is 1h and 15 min.

3.1 GIA model

In principle, there are no restrictions on the spatial and temporal features of the ice melting history that can be employed in SELEN⁴, provided that the model is properly discretized according to the scheme outlined in S8.7. Similarly, any linear rheological profile is a priori acceptable for the mantle and for the lithosphere, as long as the Love numbers can be cast in a normal-mode multi-exponential form (Peltier, 1974). Due to its central role in the context of contemporary GIA studies, in our test run we have implemented an ad hoc realization of the GIA model ICE-6G_C(VM5a), originally introduced by Peltier et al. (2015).
3.1.1 Ice melting history

Ice thickness data for model ICE-6G_C have been downloaded from the home page of Prof. WR Peltier on August 2016. The data span the last 26,000 yrs and are provided on a $1^\circ \times 1^\circ$ global cartesian latitude-longitude grid (the number of grid points is thus 64,800). In each of the grid cells, the time history of ice thickness is assumed to evolve in a piecewise linear manner, with variable increments of 1.0 or 0.5 kyrs. Thus, to fit the SELEN$^4$ default input format, we have first mapped the original thickness data on a spherical equal-area Tegmark grid described in S8.6 of SSM19. In doing that, we have chosen a resolution parameter $R = 44$, so that the grid consists of $P = 75,692$ pixels (or cells), each with a radius of $\sim 46$ km. The cells number is thus a bit larger than the number of cells in the original cartesian grid. In addition, we have transformed the original time history in a piecewise constant form with a uniform spacing of 0.5 kyrs, assuming no glaciation phase prior to deglaciation. Because of the adaptations we have made, the ice model so obtained is not an exact replica of ICE-6G_C, but a particular realization of it. Hence, to avoid any ambiguity, in the following it shall be referred to as I6G-T05-R44. Assuming an ice density $\rho_w = 931$ kg m$^{-3}$ and that the area of the oceans is fixed to the present value, I6G-T05-R44 holds 206.5 m of equivalent sea level at 26 ka and 75.1 m at present, corresponding to a total eustatic sea-level rise of 131.4 m since the inception of melting (26 ka). The ice thickness of I6G-T05-R44 for a few time frames is shown in Figure 1.

3.1.2 Present day topography

In order to reconstruct the whole history of the Earth’s topography and of Relative Sea Level (RSL) since the inception of deglaciation, it is necessary to impose the present relief as a final condition for the Sea Level Equation (see Peltier, 1994). In this test run, we have utilized the “bedrock version” of the global dataset ETOPO1 (Amante and Eakins, 2009; Eakins and Sharman, 2012) as the final condition, whereas the final condition for the ice thickness is given by the last time frame of I6G-T05-R44. ETOPO1 is distributed on a cartesian longitude-latitude grid with a resolution of 1 arc-minute ($\frac{1}{60}$°), so that an interpolation of topography on the Tegmark grid has been performed. Note that in regions where small-scale topographic features are present, such as ocean trenches, interpolating a high-resolution relief model on a coarse Tegmark grid may yield inaccurate results. In this case, a different approach should be adopted for the realization of topography, based for instance on a binning algorithm. Other choices of the final topography are of course possible. For example, in order to ensure the maximum accuracy in Antarctica, the original model ICE-6G_C employs the Bedmap2 data set of Fretwell et al. (2013) south of 60°S latitude. In SELEN$^4$, there are no restrictions on the choice of the final relief, which is left to the user. In Figure 2 we show the realization of the modern bathymetry that we have obtained by interpolating ETOPO1 on the same Tegmark grid that we have used for the ice sheets in this test run, which shall be referred to as model ETO-R44 in the following. With SELEN$^4$, other versions of this elevation model are made available, characterized by different spatial resolutions $R$.

3.1.3 Rheological profile

The 11-layer Maxwell rheological profile of the 1-D Earth employed in the test run is shown in Table 2. For each layer, values of density and of rigidity are obtained by volume-averaging the PREM (Preliminary Reference Earth Model) of Dziewonski
and Anderson (1981) while the viscosity profile is reproduced using the data available in the supporting information supplied with Peltier et al. (2015). The 90-km thick lithosphere is elastic and the core is fluid, homogeneous and inviscid. Note that since the original VM5a profile includes elastic compressibility and reproduces the finely layered PREM structure (Peltier et al., 2015), in the following we shall refer to the model in Table 2 as VM5i. As it includes $N_v = 9$ Maxwell layers in the mantle, characterized by distinct properties, for any given harmonic degree $l$, the loading Love numbers (LLNs) and tidal Love numbers (TLNs) for model VM5i are described by a spectrum of $4N_v = 36$ viscoelastic normal modes (see e.g., Spada et al., 2011). Since we are not aware of published, community-agreed sets of Love numbers for a multi-layered compressible viscoelastic model, in the test run we rest on an incompressible profile, for which agreed results have been obtained by Spada et al. (2011). We remark, however, that SELEN$^4$ can work with compressible or transient rheologies as well, provided that LLNs and TLNs in normal-mode form (Wu and Peltier, 1982) are accessible to the user. Figures 3a and 3b show the elastic and fluid values of the LLNs in the range of harmonic degrees $1 \leq l \leq 1,024$ for model VM5i. The Love numbers are given in a geocentric reference frame with the origin in the Earth’s center of mass (CM). Numerical values of a few relevant LLNs and TLNs are listed in Table 3 and in its caption. They have been computed by the Love numbers calculator TABOO (see Spada et al., 2008).

3.2 GIA in the past

This section is devoted to the description of some outputs of the SELEN$^4$ test run, concerning the effects of GIA during the whole period after the LGM. These include i) the predictions of the history of RSL at specific sites, ii) the time evolution of paleo-topography in some regions of interest, and iii) the excursions of the Earth’s pole of rotation forced by GIA.

3.2.1 Relative Sea Level curves

Figure 4 shows data (with error bars) and SELEN$^4$ predictions for a small subset of the 392 sites contained in the RSL database of Tushingham and Peltier (1993) (hereafter referred to as TP93). In view of its historical importance in the development of GIA studies (e.g., Tushingham and Peltier, 1991, 1992; Melini and Spada, 2019), the TP93 database is available with the SELEN$^4$ package; however, there are no restrictions on the use of other datasets, or simply individual RSL records, if available to the user. Note that ages given in the TP93 database included with SELEN$^4$ are expressed in radiocarbon years and have not been calibrated to calendar years. In Figure 4, three different configurations, i.e., combinations of parameters $R$, $l_{\text{max}}$, $n_{\text{ext}}$ and $n_{\text{int}}$ have been adopted, always assuming $n_{\text{ext}} = n_{\text{int}}$. Results obtained for the three configurations are shown by different colours. The black curves have been obtained using the GIA model described in Section 3.1, characterized by the settings $R44/L128/I3$ ($R = 44$, $l_{\text{max}} = 128$, $n_{\text{ext}} = n_{\text{int}} = 3$). Blue curves have been obtained by configuring SELEN$^4$ with the combination of parameters $R100/L512/I5$ i.e., increasing the spatial resolution and the number of internal and external iterations in a significant way (a truncation degree $l_{\text{max}} = 256$ is often employed in GIA modeling, see e.g., Kendall et al. 2005). With this configuration, the pixel radius is reduced to $\sim 20$ km (see S8.6 and Table S6). Of course, the execution time of SELEN$^4$ increases significantly with respect to the test run, requiring 2.5 days ($\sim 60$ h) on a 56-core Intel Xeon E7 “Broadwell” system. It is apparent that this high-resolution case is providing results substantially matching
those of the standard run with $R44/L128/I3$. Minor differences can be noted in the early stages of deglaciation, which however do not exceed the typical uncertainty on the observed RSL values. These differences are likely to be caused by the significant changes that the topography undergoes in this early phase in the polar regions, which are better captured by increasing the model resolution. Finally, red curves have been obtained for a low resolution run with $R30/L64/I2$, whose execution time is 15 min on a 12-core Mid-2012 Mac Pro. The curves clearly indicate that computationally inexpensive runs can provide reliable results in the far field of the previously glaciated areas (e.g., in sites 639 and 525), but in the near field (e.g., site 283) they can diverge significantly from both high- and intermediate-resolution results.

From a visual inspection of Figure 4, it is apparent that at some sites the best GIA predictions fit very well the observations, like at sites 101 and 283. For others, the trend of the RSL data is captured satisfactorily (see sites 155, 209 and 328) while in some others the fit is quite poor (sites 639, 525 and 570). The identification of the possible sources of the evidenced misfits, which should be measured using rigorous statistical methods after a proper calibration of the ages, is not the purpose of this work. We only note that they do not necessarily stem from limitations of the GIA model adopted, since it is well known that at a specific site tectonic deformations can have an important role (see e.g., Antonioli et al., 2009, for a significant example) and these are not taken into account when solving the SLE. Similarly, in our formulation of the SLE we are neglecting the possible effects from the loads exerted by sediments (Dalca et al., 2013). Since program SELEN$^4$ is open source, the users can modify the code to account for non glacial loads and change the configuration to determine more suitable combinations of the basic ingredients of GIA modeling i.e., the history of deglaciation and the rheological layering of the mantle, in order to improve the fit between model predictions and any preferred RSL dataset.

### 3.2.2 Paleo-topography

SELEN$^4$ allows for a gravitationally and topographically self-consistent description of the evolution of sea level, along the route highlighted by Peltier (1994) and Lambeck (2004). This implies that SELEN$^4$ can iteratively reconstruct the time evolution of the coastlines and of the OF, in a fashion that is consistent with the gravitational, rotational and deformational effects induced by deglaciation (for a full account of the theory, the reader is referred to SSM19). These features make the SLE an integral 3-D non-linear equation (Spada, 2017). The importance of the evolution of paleo-topography for the development of human culture since the LGM has been pointed out in a number of works (see e.g., Cavalli-Sforza et al., 1993; Peltier, 1994; Lambeck, 2004; Dobson, 2014, and references therein). Recently, in the context of GIA modeling, Spada and Galassi (2017) have faced the problem of the dynamic evolution of aquaterra, i.e., the land that has been inundated and exposed during the last glacial cycle (Dobson, 1999, 2014), using the same approach adopted in this work.

A standard SELEN$^4$ output is shown in Figure 5, where the Earth’s relief at the LGM is reconstructed in the post-processing phase of SELEN$^4$, according to the solution of the SLE. Note that the figure only shows the bedrock relief at the LGM, which is not what Peltier (1994) has called true paleo-topography (PT), which also includes the contribution of ice elevation. The user can easily obtain maps of the full PT merging maps like that shown in Figure 5 with those of Figure 1. **Light blue areas across the polar regions covered by thick ice at the LGM correspond to places where the ice was grounded below sea level at that epoch. In SELEN$^4$, it is also possible to visualize the full time evolution of the OF, in order to better appreciate, for**
example, the spatiotemporal distribution of the ice shelves, which cannot be seen in this paleo-topography map. At the
global scale, the major land masses that were exposed at the LGM are clearly visible, as evidenced by the low elevation
areas; these include Beringia, Sundaland, Patagonia, Sahul, Doggerland, and the East Siberian shelf. Using GIA
models, the time evolution of these land masses has been investigated in a number of papers, also in view of the
important implications on the dispersal of modern humans. For specific case studies, the reader is referred to the works
of Peltier and Drummond (2002), Lambeck (2004), Klemann et al. (2015), Spada and Galassi (2017), and to references
therein.

By increasing the spatial resolution, SELEN$^4$ can also be employed to resolve the past sea-level variations on regional
scale. As a specific case study, we consider the Mediterranean Sea. The history of RSL in the Mediterranean Sea has been the
subject of various investigations, stimulated by the amount of high-quality geological, geomorphological and archaeological
indicators in the region (see e.g., Lambeck and Purcell, 2005; Antonioli et al., 2009; Evelpidou et al., 2012; Vacchi et al., 2016;
Roy and Peltier, 2018, and references therein). Since on the global scale of Figure 5 the details of the paleo-topography in this
area are difficult to visualize, we have used the outputs of the high-resolution run with settings $R100/L512/I5$, already exploited
in Section 3.2.1. The results are shown in the map of Figure 6, where paleo-topography is shown at 26 ka. The vastly exposed
continental shelf of Tunisia (Mauz et al., 2015) and the northern Adriatic Sea (Lambeck and Purcell, 2005) are now clearly
visible, along with other smaller scale regions where the topography has seen significant changes during the last deglaciation
(Lambeck, 2004; Purcell et al., 2007).

### 3.2.3 Polar motion

With SELEN$^4$, three configurations are possible, in which rotational effects on GIA are dealt with in different manners. First,
these effects can be simply ignored, as it is done in the classical FC76 GIA theory. However, when rotational effects are taken
into consideration, this can be done in two different ways, i.e., either following the traditional rotation theory (Milne and
Mitrovica, 1998; Spada et al., 2011) or a revised rotation theory proposed by Mitrovica et al. (2005) and Mitrovica and Wahr
(2011). The reader is referred to the literature for a detailed presentation of the two theories and to S5.2 for a brief account. We
also refer to Martinec and Hagedoorn (2014) for a general formulation dealing with rotational effects in GIA modeling.

Here it is useful to mention that in the traditional treatment the long-term response of the Earth is evaluated assuming that
the lithosphere is characterized by a finite elastic strength, while in the revised theory the equilibrium rotational shape is,
more realistically, only based on the viscous properties of the planet. Furthermore, the long term extra-flattening due to mantle
dynamics is properly accounted for. We remark that in both cases the fast Chandler wobble component of polar motion is
filtered out since the onset from the Liouville equations, because the time scales of GIA largely exceed the Chandler wobble
period (≈ 14 months, see e.g., Lambeck 1980). The implications for the GIA response of the new theory are quite significant,
as illustrated in detail by Mitrovica et al. (2005) and Mitrovica and Wahr (2011) and shown in Figure 7.

Solid curves in Figure 7 show the evolution of the polar motion components $(m_x, m_y)$ and their rates of change $(\dot{m}_x, \dot{m}_y)$
in response to GIA, obtained by solving the Liouville equations in the test run, in which the revised rotation theory is em-
ployed. Dashed curves show results obtained with the traditional theory. As the $x$ and $y$ components of polar motion are
conventionally measured along the Greenwich meridian and $90^\circ$E, respectively. Figure 7a indicates that since the inception of deglaciation, the displacement of the pole has been roughly in the direction of the Hudson Bay, consistent with the seminal results of Sabadini and Peltier (1981). The two theories predict similar evolutions of the pole of rotation, which according to Figure 7a has been displaced by $\sim 18$ km on the Earth surface by the glacial readjustment process since 26 ka. We note that at the time of the rapid melting episode known as Melt Water Pulse MWP-1A (between 14.3 and 12.8 ka, see e.g., Blanchon 2011) a sudden variation in the $m_y$ component of polar motion has occurred but no changes can be observed on $m_x$. When the \textit{rates} of polar motion are considered in Figure 7b, differences between the predictions of the two rotations theories are more apparent. In particular, the present-day (0 ka) \textit{rates} of polar motion are found to be $\sim 1$ and $\sim 3$ deg Ma$^{-1}$ for the revised and the traditional theories, respectively, which fits the predictions of Mitrovica and Wahr (2011) and confirms that the traditional theory largely overestimates the effects of GIA on polar motion. We note also that MWP-1A has caused a remarkable acceleration of polar motion, with a variation in the rate of $\sim 4$ deg Ma$^{-1}$ during a few centuries. As far as we know, the \textbf{possible} geophysical consequences of such sudden acceleration in the pole path have not been investigated yet.

### 3.3 GIA at present

In this section we describe further outputs of the SELEN$^4$ test run, focussing in the effects of GIA at present time. In particular, we consider \textit{i)} the global pattern of the so-called \textit{GIA fingerprints}, \textit{ii)} predictions of the the rate of sea-level change at tide gauges, and \textit{iii)} the time-variations of the Stokes coefficients of the Earth’s gravity field induced by GIA.

#### 3.3.1 GIA fingerprints

Figure 8 shows another standard output of SELEN$^4$, \textit{i.e.}, the present-day rates of variation of four fundamental quantities associated with GIA, obtained for the test run. These are \textbf{relative sea level change} ($\dot{S}$, frame a), vertical displacement of the crust ($\dot{U}$, frame b), absolute sea level ($\dot{N}$, frame c) and \textbf{geoid height} ($\dot{G}$, frame d). \textbf{Generalizing the definition of sea-level fingerprint} introduced by Plag and Jüettner (2001) in the context of contemporary ice mass changes, these have been referred to as \textit{GIA fingerprints} by Spada and Melini (2019). The spatial variability of the GIA fingerprints reflect the effects of deformation, gravitational attraction, and rotation within the system composed by the solid Earth, the oceans and the ice sheets (Clark et al., 1978; Mitrovica and Milne, 2002). In view of their importance on the interpretation of ground-based (King et al., 2010) or satellite geodetic observations (Peltier, 2004) and of \textbf{tide gauge} secular trends \textit{(e.g., Spada and Galassi, 2012; Wöppelmann and Marcos, 2016)}, their properties have been the subject of various investigations during last decade \textit{(see e.g., Mitrovica et al., 2011; Tamisiea, 2011; Spada and Galassi, 2015; Spada, 2017; Husson et al., 2018; Melini and Spada, 2019; Spada and Melini, 2019)}.

It should be remarked that the four fingerprints shown in Figure 8 are not independent of one another. In particular, the SLE gives $\dot{S} = \dot{N} - \dot{U}$ according to Eq. (21). Furthermore, $\dot{N} = \dot{G} + \dot{c}$, where $c$ is \textbf{the FC76 constant} \textit{(see S2.4 in SSM19)}. The two relationships above hold \textbf{regardless of the} particular combination of rheology and ice model employed, and the preferred rotation theory adopted. However, the patterns of the \textbf{GIA fingerprints} and the numerical value of $\dot{c}$ are model-dependent. Other interesting results hold for the spatial averages of the fingerprints, which reflect some physical aspects of GIA (Spada,
2017; Spada and Melini, 2019) and are useful to correct geodetic observations from the effects of deglaciation (e.g., Spada and Galassi, 2015). In Table 4 we summarize the numerical values of whole Earth surface averages (denoted by symbol $< \cdots >^e$) and ocean-averages ($< \cdots >^o$) of the GIA fingerprints in the test run. In addition, we have also executed SELEN$^4$ adopting the traditional rotation theory and neglecting rotational effects, and the corresponding averages are shown in Table 4 as well. We note that by virtue of mass conservation $< \dot{\mathcal{G}} >^e = < \dot{U} >^e = 0$ (see S4.3 and S6.2) regardless of the rotation theory adopted, and as a consequence $< \dot{S} >^e = < \dot{N} >^e = \dot{c}$. We also note that the numerical value of $< \dot{N} >^o$, commonly employed to correct the altimetry observations of absolute sea-level change for the effects of GIA, is in fair agreement with predictions from state-of-the-art GIA models (e.g., Church et al., 2013a; Spada and Galassi, 2015; Spada, 2017). Notably, $< \dot{N} >^o$ is not very sensitive to the choice of the rotation theory. Since model I6G-T05-R44 assumes that melting of the major ice sheets ceased $\sim 4,000$ years ago, the small value of $< \dot{S} >^o$ only reflects ongoing changes in the area of the oceans due to GIA. In the FC76 fixed-shorelines approximation, $< \dot{S} >^o$ would be identically zero by virtue of the mass conservation principle (see e.g., Spada, 2017; Spada and Melini, 2019).

### 3.3.2 GIA at tide gauges

Estimating global mean sea-level rise in response to climate change requires to correct tide gauge records for the effects of GIA. Since the late 1980s, with the awareness of global warming and the availability of numerical solutions of the SLE (Peltier and Tushingham, 1989), GIA corrections to the observed trends of sea level have been routinely applied (for a review, see Spada and Galassi, 2012; Spada et al., 2015; Wöppelmann and Marcos, 2016). However, since GIA models are progressively improved to provide a better description of reality, corrections at tide gauges are not given once and for all (Kendall et al., 2006; Tamisiea, 2011; Melini and Spada, 2019). Furthermore, new constraints from past sea level or modern geodetic observations have permitted gradual refinements (either by formal inverse methods or simply by trial and error) of the two basic ingredients of GIA modeling, i.e., the Earth rheological profile and the history of deglaciation since the LGM. Uncertainties in modeling are significant (Melini and Spada, 2019), which constitutes an additional motivation to improve the approach to GIA.

In Table 5, we show SELEN$^4$ predictions for $\dot{S}$ at a few tide gauges, for the test run and other possible configurations as well. Here we only show results for the 23 sites that have been considered by Douglas (1997) in his redetermination of global sea-level rise, which obey specific criteria that make them suitable to represent the trend of secular sea-level rise. The sites chosen by Douglas (1997) are located in the periphery of the regions covered by thick ice sheets at the LGM, since GIA predictions at sites formerly beneath the ice sheets are expected to be more affected by uncertainties in GIA modeling. This has been recently confirmed by Melini and Spada (2019). The post-processing phase of SELEN$^4$ can be configured to handle any properly formatted input dataset with coordinates of geodetic points of interest, where all the variables considered in Figure 8 can be evaluated. This can be useful, for instance, to estimate the effects of GIA on vertical movements at specific GPS points (Serpelloni et al., 2013). Modules for the computation of horizontal displacements shall be included in future releases. Comparing column $d)$ with $b)$ and $c)$ (all these runs are characterized by the intermediate resolution configuration $R44/L128/I3$), we note that rotational effects are important at tide gauges; however from $b)$ and $c)$ we also note that differences between the revised and the traditional rotation theory generally do not exceed the 0.1 mm yr$^{-1}$ level at the tide gauges.
considered here. Comparing outputs in b) with the high-resolution run in a), we note maximum differences of 0.03 mm yr$^{-1}$, which further confirms the reliability of the test run.

Comparing the high-resolution results in Table 5 ($R100/L512/I5$, column a)) with those obtained using the ICE-6G_C(VM5a) model in the original implementation of WR Peltier (see http://www.atmosp.physics.utoronto.ca/~peltier/data.php, last accessed 06 June 2019), reported in column e), we note a fair agreement between the two model predictions. The two sets of predictions have the same sign with the only exception of Cascais and Lagos. The differences between the values obtained with $R100/L512/I5$ and the original implementation ICE-6G_C(VM5a) are generally close to 0.2 mm yr$^{-1}$ or slightly larger. The values of $\dot{S}$ averaged over the tide gauges differ, but they are both $\leq 0.1$ mm yr$^{-1}$. More significant differences in the $\dot{S}$ values are however apparent when we compare model predictions for sites located in the polar regions beneath the former ice sheets, which are not considered in Table 5. At these locations, the expected $\dot{S}$ values are of the order of several mm yr$^{-1}$, due to the large isostatic disequilibrium still associated with ice unloading. For example, at the tide gauge of Stockholm, we obtain a rate of $-4.43$ mm yr$^{-1}$ for the high resolution run $R100/L512/I5$ while in the original ICE-6G_C(VM5a) implementation, the rate is $-3.75$ mm yr$^{-1}$. To better evaluate the meaning of these differences, it is worth to note that they largely exceed the typical $2\sigma$ uncertainty in the observed rates, which according to Spada and Galassi (2012) (see their Table 2), is never larger than 0.3 mm yr$^{-1}$. We have also ascertained that misfits of the order of 1 mm yr$^{-1}$ are not uncommon in other high-latitude sites of both hemispheres. Tracing the origin of the discrepancies in the two sets of GIA predictions (and therefore in the whole set of the GIA geodetic fingerprints considered in Figure 8) is not easy at this stage, and would demand a detailed model inter-comparison study like those performed in the GIA community by Spada et al. (2011) and Martinec et al. (2018). We can however guess that the misfit between the two sets of GIA predictions stems from the different discretisations of the ice time-histories, from the effects of mantle compressibility, and possibly from the different rotation theories adopted.

3.3.3 Stokes coefficients of the gravity field

In Figure 9 we study the present-day rates of change of the variations of Stokes coefficients ($\dot{\delta}c_{lm}, \dot{\delta}s_{lm}$) induced by GIA, computed in the test run with $R44/L128/I3$. These quantities represent the coefficients of the expansion of the gravity potential variation $\Phi(\gamma, t) - \Phi(\gamma, t_0)$ in series of spherical harmonics, hence they contain information upon the response of the Earth to surface loading and to movements of the axis of rotation. In SELEN$^4$ we use a real, fully normalised representation for the Stokes coefficients, following the Gravity Recovery and Climate Experiment (GRACE) conventions for spherical harmonics (see, in particular, Bettadpur 2018 and S8.9). However, it is important to note that in Figure 9, the Stokes coefficients also include the direct effect of Earth rotation, associated with the change in centrifugal potential, on the degree 2 TLNs (i.e., they account for the ‘$\delta(t)$’ term in ‘$\delta(t) + k_2^T(t)$’, see Eq. S173), hence the rates we have computed are not directly comparable with the GRACE rates. In fact, since in its orbit GRACE is not physically connected with the Earth, it cannot be influenced by the direct rotational effect (the whole issue has been the subject of discussion a few years ago, see Chambers et al., 2010; Peltier et al., 2012; Chambers et al., 2012). The user of SELEN$^4$, however, can supply the program with a rotation response.
function $G^{\text{rot}}$ that does not include the direct term in order to produce GRACE-compliant Stokes coefficients which are only indirectly affected by Earth rotation.

The fully normalised cosine (squares) and sine (circles) coefficients ($\delta c_{lm}$, $\delta s_{lm}$) are shown in Figure 9a only for harmonics with degree $l \leq 6$. The dominance of the degree 2 coefficients is apparent, which reflect the symmetries of the $\dot{G}$ fingerprint in Figure 8d. We note that since $\dot{N} = \dot{G} + \dot{c}$, where $\dot{N}$ is the absolute sea-level fingerprint in Figure 8c and $c$ is the FC76 constant (see Eq. 22), the Stokes coefficients for $\dot{G}$ and for $\dot{N}$ coincide for $l \geq 2$. For reference, the numerical values of the degree $l = 2$ coefficients obtained in the test run are $\delta c_{20} = +1.59$, $\delta c_{21} = -0.76$, $\delta s_{21} = +3.37$, $\delta c_{22} = -0.35$, and $\delta s_{22} = +0.07$ in units of $10^{-11} \text{ yr}^{-1}$; the modulus of all other coefficients is $<1$ in these units. To better visualize the decay of the Stokes coefficients with increasing $l$, in the diagram of Figure 9b we show the harmonic spectrum defined in S8.9 (see Eq. S477). By inspection of the spectrum plot it is now apparent that the energy contained in the degree $l = 2$ harmonic component exceeds by at least one order of magnitude all those with $l \geq 3$. After a plateau that indicates a substantial power equipartition in the range of harmonics $3 \leq l \leq 7$, the spectrum clearly shows a red character and decays very rapidly, closely following a power law $\sim l^{-5}$ (solid line). This result is consistent with those obtained by Spada and Galassi (2015), although they have used a simplified GIA model with fixed shorelines and the traditional rotation theory. They confirm that, for $l \geq 3$, the power contained in the GIA-induced regional variations in absolute sea level is negligible when compared with the spectrum observed during the altimetry era.

4 Conclusions

We have presented an updated version of the SLE solver SELEN, which was originally introduced by Spada and Stocchi (2007) and principally meant as a tool for students. Along with a condensed theory background and the basic features of the new program, we have provided a step-by-step description of the outcomes of a medium-resolution test run that requires modest computing resources. However, the run accounts for an up-to-date description of the time history of melting since the LGM and a realistic rheological profile, being based upon a realization of model ICE-6G_C (VM5a) of Peltier et al. (2015).

The outputs of the test run, which cover different temporal scales, have been briefly discussed in order to appreciate some of the possible geodynamical implications. Outputs of a high-resolution test run have been also presented to illustrate the effects of spatial and harmonic resolution on some GIA predictions.

With respect to the original version of the code, two major improvements have been made in SELEN$^4$. The first is represented by an increased realism in the description of the GIA process. Indeed, now the program accounts for the migration of the shorelines and for the rotational feedback on sea-level change, which enable a fully topographically and gravitationally self-consistent modelization, in the sense defined by Peltier (1994). Furthermore, SELEN$^4$ can be configured assuming two different rotation theories, or even excluding rotational effects. The second improvement is in terms of usability, efficiency and versatility, and covers various aspects. First, the solution of the SLE is now performed by a single Fortran program unit, leaving to a flexible and customizable post-processor the computation of various outputs encompassing the broad spectrum of the GIA phenomenology. Second, on modern multi-core systems, SELEN$^4$ can take advantage of multi-threaded parallelism to
speed up the most computationally intensive portions of the code. The reader can find details about computational aspects as execution time and code parallelism in Section S8.7 of SSM19. Third, the user can easily customize the time evolution of the surface load and the rheological layering of the Earth providing pre-computed loading and tidal Love numbers. Last, SELEN⁴ comes with a User guide and with a fully detailed theory background in a supplement, which is particularly meant to illustrate the basic concepts of GIA to young scientists or colleagues and to allow transparency and reproducibility.

In a recent benchmarking initiative (Martinec et al., 2018), a preliminary version of the new program has been successfully tested against other independently developed SLE solvers, in the particular case in which rotational effects are not taken into account and assuming simplified surface loads. The version of SELEN⁴ that is distributed with the present work reproduces the numerical results published in Martinec et al. (2018), if the code is configured with the parameters employed in the benchmark exercise. After it has been progressively developed in various interim versions, SELEN⁴ is now released to the GIA and to the global geodynamics community as an open source tool.

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Code and data availability. SELEN⁴ is available from Zenodo at the link https://zenodo.org/record/3339209 (DOI: 10.5281/zenodo.3339209) and from the Computational Infrastructure for Geodynamics (CIG) at github.com/geodynamics/selen. The ice history data for ICE-6G (VM5a) have been downloaded from http://www.atmosp.physics.utoronto.ca/~peltier/data.php (last accessed 20 Apr 2019). The Model ETOPO1 has been obtained from https://www.ngdc.noaa.gov/mgg/global/ (last accessed 26 Feb 2019).

Copyright statement. SELEN⁴ is released under a 3-Clause BSD License (for details, see https://opensource.org/licenses/bsd-3-cla...
Author contributions. G.S. and D.M. have both contributed to the design and implementation of the research, to the analysis of the results and to the writing of the manuscript. The supplement has been written by G.S. with the support of D.M. The code has been progressively developed by G.S. and D.M., who has edited the User Guide and the on-line version of the code.

Competing interests. The authors declare no competing interests.
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Roy, K. and Peltier, W.: Relative sea level in the Western Mediterranean basin: A regional test of the ICE-7G NA (VM7) model and a constraint on late Holocene Antarctic deglaciation, Quaternary Science Reviews, 183, 76–87, 2018.


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Table 1. Details of the configuration of the SELEN\textsuperscript{4} test run whose results are considered in Section 3, with notes and links to text and figures.
Figure 1. Ice distribution according to model I6G-T05-R44, at six different epochs since 26 ka. The maps are obtained by direct triangulation of the pixelized ice thickness data using the GMT program `pscontour`. This and the following figures are drawn using GMT scripts adapted from those which are available in the output folders of SELEN\textsuperscript{4} after execution.
Figure 2. Present day relief according to model ETO-R44 used in the test run, obtained from model ETOPO1 by bilinear interpolation on the pixels of the Tegmark grid with resolution $R = 44$, using the GMT program \texttt{grdtrack}.
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**Table 2.** Density, rigidity and viscosity profiles adopted in the rheological model VM5i, where abbreviations LT, UM, TZ and LM stand for lithosphere, upper mantle, transition zone and lower mantle, respectively. **The [R2-16] radii** $r_-$ and $r_+$ **indicate the lower and upper radii** of each layer. Some [R1-48] spectral properties of model VM5i are given in Figure 3 and in Table 3.
Figure 3. Elastic (a) and fluid (b) LLNs as a function of harmonic degree $l$ for the 11-layer rheological model VM5i employed in the test run (see Table 2). It is apparent that for this model asymptotic values are reached, in both cases, for $l$ exceeding a few hundreds. Note that in (b), where the fluid LLN for vertical displacement is normalised by $(2l + 1)$, the relationship $h_i^{L_f} \approx (2l + 1)k_i^{L_f}$ is apparent.
Table 3. Numerical values of the LLNs for the rheological model VM5i (see Table 2), for some harmonic degrees \( l \). We use the compact notation \( v_e = a \times 10^{-\nu} \) and \( v^e = v \times 10^e \), where \( v \) is any value in the table and \( e \) is an exponent. Note that, for this model, the elastic TLNs of degree \( l = 2 \) are \((k_2^{Te}, h_2^{Te}, l_2^{Te}) = (0.289^0, 0.524^0, 0.108^0)\) while the fluid values are \((k_2^{Tf}, h_2^{Tf}, l_2^{Tf}) = (0.931^0, 0.191^1, 0.514^0)\). [R1-49].

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Figure 4. RSL data (with error bars) at eight of the 392 sites of the TP93 RSL database. Black curves show the results of the standard test run with configuration R44/L128/I3, the blue ones are for the high-resolution run with R100/L512/I5 while those in red are for a low-resolution configuration with R30/L64/I2. The map shows the location of the RSL [R2-17] sites.
Figure 5. Sample SELEN\textsuperscript{4} output of the global relief at the LGM (21 ka), according to the test run with R44/L128/I3, where topography ETO-R44 of Figure 2 has been used as a final condition for the SLE.
Figure 6. Paleo-topography of the Mediterranean Sea and of the Black Sea at 26 ka, obtained by a high-resolution SELEN run with configuration R100/L512/I5, in order to enlighten the exposed lands in detail.
Figure 7. Cartesian components of polar motion along the axes \( x \) and \( y \) (a) and their time derivatives (b) as a function of time since the beginning of deglaciation, for the test run with configuration R44/L128/I3. The \( x \)-axis points along the prime meridian and the \( y \) points to 90° East. Dashed and solid curves show results for the traditional and for the revised rotational theories, respectively. The steep change in the \( y \) components at \( \sim 14 \) ka is forced by the inertia variations due to the occurrence of the Melt Water Pulse MWP-1A.
Figure 8. Present-day GIA fingerprints obtained for the test run with R44/L128/I3. Note that the color table is saturated in a narrow interval. The effects of Earth rotation, evaluated according to the revised rotation theory, can be well discerned for \( \dot{N} \) and \( \dot{\mathcal{G}} \), with the characteristic high-amplitude lobes with a harmonic degree \( l = 2 \) and order \( m = 1 \) symmetry (Spada and Galassi, 2015). Spatial averages of these maps are given in Table 4. Min/max values for \( \dot{S} \), \( \dot{U} \), \( \dot{N} \), and \( \dot{\mathcal{G}} \) after truncation of the SELEN outputs to one decimal place, [R1-50] are: \(-13.3/4.8, -5.2/14.1, -0.7/1.2, \) and \(-0.5/1.4 \) in units of mm yr\(^{-1} \), respectively.
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<td>$-0.23$</td>
</tr>
<tr>
<td>$&lt; \dot{G} &gt;^o$</td>
<td>$-0.05$</td>
<td>$-0.08$</td>
<td>$-0.03$</td>
</tr>
</tbody>
</table>

Table 4. Whole Earth and ocean averages of the GIA fingerprints according to the test run based upon the new rotation theory (column a), and the traditional theory b). In c) we also consider the case when no rotational effects are taken into account. In all the computations we have adopted the combination R44/L128/I3. Note that $< \dot{N} >^e = c$, where $c$ is the FC76 constant. In this table, the SELEN$^4$ outputs are rounded to two significant figures.
Table 5. Present-day rates of sea-level change at the 23 Douglas (1997) tide gauges, for the test run of SELEN⁴ (column b)) and for other configurations. Results based upon the original implementation of model \textit{ICE-6G}_C(VM5a) [R3-4] are reproduced in column e). The average rate is also shown in the bottom line. The SELEN⁴ outputs have been rounded to two significant figures.
Figure 9. **Present-day rate of change of the low harmonic degree** \( (l \leq 6) \) Stokes coefficients obtained for the test run of SELEN\(^4\) (frame a) and their full spectrum extended to harmonic degree \( l_{\text{max}} = 128 \) (b). The solid line in (b) represents the power law that best-fits the spectrum (in the least squares sense), obtained for \( l \geq 10 \).